Microeconomic Theory

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Part I

Individual Decision Making

A distinctive feature of microeconomic theory is that it aims to model economic activity as an interaction of individual economic agents pursuing their private interests. It is therefore appropriate that we begin our study of microeconomic theory with an analysis of individual decision making.

Chapter 1 is short and preliminary. It consists of an introduction to the theory of individual decision making considered in an abstract setting. It introduces the decision maker and her choice problem, and it describes two related approaches to modeling her decisions. One, the *preference-based approach*, assumes that the decision maker has a preference relation over her set of possible choices that satisfies certain rationality axioms. The other, the *choice-based approach*, focuses directly on the decision maker's choice behavior, imposing consistency restrictions that parallel the rationality axioms of the preference-based approach.

The remaining chapters in Part One study individual decision making in explicitly economic contexts. It is common in microeconomics texts-and this text is no exception-to distinguish between two sets of agents in the economy: *individual consumers* and *firms*. Because individual consumers own and run firms and therefore ultimately determine a firm's actions, they are in a sense the more fundamental element of an economic model. Hence, we begin our review of the theory of economic decision making with an examination of the consumption side of the economy.

Chapters 2 and 3 study the behavior of consumers in a market economy. Chapter 2 begins by describing the consumer's decision problem and then introduces the concept of the consumer's *demand function*. We then proceed to investigate the implications for the demand function of several natural properties of consumer demand. This investigation constitutes an analysis of consumer behavior in the spirit of the choice-based approach introduced in Chapter 1.

In Chapter 3, we develop the classical preference-based approach to consumer demand. Topics such as utility maximization, expenditure minimization, duality, integrability, and the measurement of welfare changes are studied there. We also discuss the relation between this theory and the choice-based approach studied in Chapter 2.

In economic analysis, the aggregate behavior of consumers is often more important than the behavior of any single consumer. In Chapter 4, we analyze the extent to which the properties of individual demand discussed in Chapters 2 and 3 also hold for aggregate consumer demand.

In Chapter 5, de study the behavior of the firm. We begin by posing the firm:s decision problem, introducing its technological constraints and the assumption of profit maximization. A rich theory, paralleling that for consumer demand, emerges. In an important sense, however, this analysis constitutes a first step because it takes the objective of profit maximization as a maintained hypothesis. In the last section of the chapter, we comment on the circumstances under which profit maximization can be derived as the desired objective of the firm's owners.

Chapter 6 introduces risk and uncertainty into the theory of individual decision making. In most economic decision problems, an individual's or firm's choices do not result in perfectly certain outcomes. The theory of decision making under uncertainty developed in this chapter therefore has wide-ranging applications to economic problems, many of which we discuss later in the book.

Chapter 1
Preference and Choice

1.A Introduction

In this chapter, we begin our study of the theory of individual decision making by considering it in a completely abstract setting. The remaining chapters in Part I develop the analysis in the context of explicitly economic decisions.

The starting point for any individual decision problem is a *set of possible (mutually exclusive) alternatives* from which the individual must choose. In the discussion that follows, we denote this set of alternatives abstractly by X. For the moment, this set can be anything. For example, when an individual confronts a decision of what career path to follow, the alternatives in X might be: {go to law school, go to graduate school and study economics, go to business school,...,become a rock star}. In Chapters 2 and 3, when we consider the consumer's decision problem, the elements of the set X are the possible consumption choices.

There are two distinct approaches to modeling individual choice behavior. The first, which we introduce in Section 1.B, treats the decision maker's tastes, as summarized in her *preference relation*, as the primitive characteristic of the individual. The theory is developed by first imposing rationality axioms on the decision maker's preferences and then analyzing the consequences of these preferences for her choice behavior (i.e., on decisions made). This preference-based approach is the more traditional of the two, and it is the one that we emphasize throughout the book.

The second approach, which we develop in Section 1.C, treats the individual's choice behavior as the primitive feature and proceeds by making assumptions directly concerning this behavior. A central assumption in this approach, the *weak axiom of revealed preference*, imposes an element of consistency on choice behavior, in a sense paralleling the rationality assumptions of the preference-based approach. This choice-based approach has several attractive features. It leaves room, in principle, for more general forms of individual behavior than is possible with the preference-based approach. It also makes assumptions about objects that are directly observable (choice behavior), rather than about things that are not (preferences). Perhaps most importantly, it makes clear that the theory of individual decision making need not be based on a process of introspection but can be given an entirely behavioral foundation.

Understanding the relationship between these two different approaches to modeling individual behavior is of considerable interest. Section 1.D investigates this question, examining first the implications of the preference-based approach for choice behavior and then the conditions under which choice behavior is compatible with the existence of underlying preferences. (This is an issue that also comes up in Chapters 2 and 3 for the more restricted setting of consumer demand.)

For an in-depth, advanced treatment of the material of this chapter, see Richter (5, 1971).

1.B Preference Relations

In the preference-based approach, the objectives of the decision maker are summarized in a *preference relation*, which we denote by $\succsim$. Technically, $\succsim$ is a binary relation on the set of alternatives X, allowing the comparison of pairs of alternatives $x, y \in X$. We read $x \succsim y$ as "x is at least as good as y." From $\succsim$, we can derive two other important relations on X:

(i) The *strict preference* ration, $\succ$, defined by

$$x \succ y \Leftrightarrow x \succsim y \text{ but not } y \succsim x$$

and read "x is preferred to y". [1]

(ii) The *indifference* relation, $\sim$, defined by

$$x \sim y \Leftrightarrow x \succsim y \text{ and } y \succsim x$$

and read "x is indifferent to y".

In much of microeconomic theory, individual preferences are assumed to be *rational*. The hypothesis of rationality is embodied in two basic assumptions about the preference relation $\succsim$: *completeness and transitivity*.[2]

Definition 1.B.1. The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness*: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

The assumption that $\succsim$ is complete says that the individual has a well-defined preference between any two possible alternatives. The strength of the completeness assumption should not be underestimated. Introspection quickly reveals how hard it is to evaluate alternatives that are far from the realm of common experience. It takes work and serious reflection to find out one's own preferences. The completeness axiom says that this task has taken place: our decision makers make only meditated choices.

Transitivity is also a strong assumption, and it goes to the heart of the concept of rationality. Transitivity implies that it is impossible to face the decision maker with a sequence of pairwise choices in which her preferences appear to cycle: for example, feeling that an apple is at least

[1] The symbol $\leftrightarrow$ is read as "if and only if." The literature sometimes speaks of $x \succsim y$ as "x is weakly preferred to y" and $x \succ y$ as "x is strictly preferred to y." We shall adhere to the terminology introduced above.

[2] Note that there is no unified terminology in the literature; *weak order* and *complete preorder* are common alternatives to the term *rational preference relation*. Also, in some presentations, the assumption that $\succsim$ is *reflexive* (defined as $x \succsim x$ for all $x \in X$) is added to the completeness and transitivity assumptions. This property is, in fact, implied by completeness and so is redundant.

as good as a banana and that a banana is at least as good as an orange but then also preferring an orange over an apple. Like the completeness property, the transitivity assumption can be hard to satisfy when evaluating alternatives far from common experience. As compared to the completeness property, however, it is also more fundamental in the sense that substantial portions of economic theory would not survive if economic agents could not be assumed to have transitive preferences.

The assumption that the preference relation $\succsim$ is complete and transitive has implications for the strict preference and indifference relations $\succ$ and $\sim$. These are summarized in Proposition 1.B.1, whose proof we forgo. (After completing this section, try to establish these properties yourself in Exercises 1.D and 1.D.)

Proposition 1.B.1. If $\succsim$ is rational then:

(i) $\succ$ is both *irreflexive* ($x \succ x$ never holds) and *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ z$).

(ii) $\sim$ is *reflexive* ($x \sim x$ for all x), *transitive* (if $x \sim y$ and $y \sim z$, then $x \sim z$), and *symmetric* (if $x \sim y$, then $y \sim x$).

(iii) if $x \sim y \succsim z$, then $x \succ z$.

The irreflexivity of $\succ$ and the reflexivity and symmetry of $\sim$ are sensible properties for strict preference and indifference relations. A more important point in Proposition 1.B.1 is that rationality of $\succsim$ implies that both $\succ$ and $\sim$ are transitive. In addition, a transitive-like property also holds for $\succ$ when it is combined with an at-least-as-good-as relation, $\succsim$.

An individual's preferences may fail to satisfy the transitivity property for a number of reasons. One difficulty arises because of the problem of *just perceptible differences*. For example, if we ask an individual to choose between two very similar shades of gray for painting her room, she may be unable to tell the difference between the colors and will therefore be indifferent. Suppose now that we offer her a choice between the lighter of the two gray paints and a slightly lighter shade. She may again be unable to tell the difference. If we continue in this fashion, letting the paint colors get progressively lighter with each successive choice experiment, she may express indifference at each step. Yet, if we offer her a choice between the original (darkest) shade of gray and the final (almost white) color, she would be able to distinguish between the colors and is likely to prefer one of them. This, however, violates transitivity.

Another potential problem arises when the manner in which alternatives are presented matters for choice. This is known as the framing problem. Consider the follow'ing example, paraphrased from Kahneman and Tversky (3, 1984):

> Imagine that you are about to purchase a stereo for 125 dollars and a calculator for 15 dollars. The salesman tells you that the calculator is on sale for 5 dollars less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?

It turns out that the fraction of respondents saying that they would travel to the other store for the 5 dollar discount is much higher than the fraction who say they would travel when the question is changed so that the 5 dollar saving is on the stereo. This is so even though the ultimate saving obtained by incurring the inconvenience of travel is the same in both cases.[3] Indeed, we would expect indifference to be the

[3] Kahneman and Tversky attribute this finding to individuals keeping "mental accounts" in which the savings are compared to the price of the item on which they are received.

response to the following question:

> Because of a stockout you must travel to the other store to get the two items, but you will receive 5 dollars off on either item as compensation. Do you care on which item this 5 dollar rebate is given?

If so, however, the individual violates transitivity. To see this, denote

$x =$ Travel to the other store and get a 5 dollar discount on the calculator.

$y =$ Travel to the other store and get a 5 dollar discount on the stereo.

$z =$ Buy both items at the first store.

The first two choices say that $x \succ z$ and $z \succ y$, but the last choice reveals $x \sim y$. Many problems of framing arise when individuals are faced with choices between alternatives that have uncertain outcomes (the subject of Chapter 6). Kahneman and Tversky (1984) provide a number of other interesting examples.

At the same time, it is often the case that apparently intransitive behavior can be explained fruitfully as the result of the interaction of several more primitive rational (and thus transitive) preferences. Consider the following two examples

(i) A household formed by Mom (M), Dad (D), and Child (C) makes decisions by majority voting. The alternatives for Friday evening entertainment are attending an opera (O), a rock concert (R), or an ice-skating show (I). The three members of the household have the rational individual preferences: $O \succ_M R \succ_M I$, $I \succ_D O \succ_D R$, $R \succ_C I \succ_C O$, where $\succ_M$, $\succ_D$, $\succ_C$, are the transitive individual strict preference relations. Now imagine three majority-rule votes: O versus R, R versus I, and I versus O. The result of these votes (O will win the first, R the second, and I the third) will make the household's preferences $\succsim$ have the intransitive form: $O \succ R \succ I \succ O$. (The intransitivity illustrated in this example is known as the *Condorcet paradox*, and it is a central difficulty for the theory of group decision making. For further discussion, see Chapter **??**.)

(ii) Intransitive decisions may also sometimes be viewed as a manifestation of a change of tastes. For example, a potential cigarette smoker may prefer smoking one cigarette a day to not smoking and may prefer not smoking to smoking heavily. But once she is smoking one cigarette a day, her tastes may change, and she may wish to increase the amount that she smokes. Formally, letting y be abstinence, x be smoking one cigarette a day, and z be heavy smoking, her initial situation is y, and her preferences in that initial situation are $x \succ y \succ z$. But once x is chosen over y and z, and there is a change of the individual's current situation from y to x, her tastes change to $z \succ x \succ y$. Thus, we apparently have an intransitivity: $z \succ x \succ z$. This *change-of-tastes* model has an important theoretical bearing on the analysis of addictive behavior. It also raises interesting issues related to commitment in decision making [see Schelling (7, 1979). A rational decision maker will anticipate the induced change of tastes and will therefore attempt to tie her hand to her initial decision (Ulysses had himself tied to the mast when approaching the island of the Sirens).

It often happens that this change-of-tastes point of view gives us a well-structured way to think about *nonrational* decisions. See Elster (2, 1979) for philosophical discussions of this and similar points.

Utility Functions

In economics, we often describe preference relations by means of a *utility function*. A utility function $u(x)$ assigns a numerical value to each element in X, ranking the elements of X in accordance with the individual's preferences. This is stated more precisely in Definition **??**.

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a *utility function representing preference relation* $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \succsim u(y)$$

Note that a utility function that represents a preference relation $\succsim$ is not unique. For any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preferences as $u(\cdot)$; see Exercise 1.D. It is only the ranking of alternatives that matters. Properties of utility functions that are invariant for any strictly increasing transformation are called *ordinal*. *Cardinal* properties are those not preserved under all such transformations. Thus, the preference relation associated with a utility function is an ordinal property. On the other hand, the numerical values associated with the alternatives in X, and hence the magnitude of any differences in the utility measure between alternatives, are cardinal properties.

The ability to represent preferences by a utility function is closely linked to the assumption of rationality. In particular, we have the result shown in Proposition 1.B.2.

Proposition 1.B.2. A preference relation $\succsim$ can be represented by a utility function only if it is rational.

Proof. To prove this proposition, we show that if there is a utility function that represents preferences $\succsim$,then $\succsim$ must be complete and transitive.

Completeness. Because $u(\cdot)$ is a real-valued function defined on X, it must be that for any $x, y \in X$, either $u(x) \succsim u(y)$ or $u(y) \succsim u(x)$. But because $u(\cdot)$ is a utility function representing $\succsim$, this implies either that $x \succsim y$ or that $y \succsim x$ (recall Definition 1.B.2). Hence, $\succsim$ must be complete.

Transitivity. Suppose that $x \succsim y$ and $y \succsim z$. Because $u(\cdot)$ represents $\succsim$, we must have $u(x) \succsim u(y)$ and $u(y) \succsim u(z)$. Therefore, $u(x) \succsim u(z)$. Because $u(\cdot)$ represents $\succsim$, this implies $x \succsim z$. Thus, we have shown that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, and so transitivity is established. *Q.E.D*

At the same time, one might wonder, can any rational preference relation $\succsim$ be described by some utility function? It turns out that, in general, the answer is no. An example where it is not possible to do so will be discussed in Section **??**. One case in which we can always represent a rational preference relation with a utility function arises when X is finite (see Exercise 1.D). More interesting utility representation results (e.g., for sets of alternatives that are not finite) will be presented in later chapters.

1.C Choice Rules

In the second approach to the theory of decision making, choice behavior itself is taken to be the primitive object of the theory. Formally, choice behavior is represented by means of a *choice structure*. A choice structure $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) a is a family (a set) of nonempty subsets of X; that is, every element of $\mathscr{B}$? is a set $B \subset X$. By analogy with the consumer theory to be developed in Chapters 2 and 3, we call the elements $B \in \mathscr{B}$ *budget sets*. The budget sets in $\mathscr{B}$ should be thought of as an exhaustive listing of all the choice experiments that the institutionally, physically, or otherwise restricted social situation can conceivably pose to the decision maker. It need not, however, include all possible subsets of X. Indeed, in the case of consumer demand studied in later chapters, it will not.

(ii) $C(\cdot)$ is a *choice rule* (technically, it is a correspondence) that assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathscr{B}$?. When $C(B)$ contains a single element, that element is the individual's choice from among the alternatives in B. The set $C(B)$ may, however, contain more than one element. When it does, the elements of $C(B)$ are the alternatives in B that the decision maker *might* choose; that is, they are her *acceptable alternatives* in B. In this case, the set $C(B)$ can be thought of as containing those alternatives that we would actually see chosen if the decision maker were repeatedly to face the problem of choosing an alternative from set B.

Example 1.C.1. Suppose that $X = \{x,y,z\}$ and $\mathscr{B} = \{\{x,y\},\{x,y,z\}\}$. One possible choice structure is $(\mathscr{B}, C_1(\cdot))$, where the choice rule $C_1(\cdot)$ is: $C_1(\{x,y\}) = \{x\}$ and $C_1(\{x,y,z\}) = \{x\}$. In this case, we see x chosen no matter what budget the decision maker faces.

Another possible choice structure is $(\mathscr{B}, C_2(\cdot))$, where the choice rule $C_2(\cdot)$ is: $C_2(\{x,y\}) = \{x\}$ and $C_2(\{x,y,z\}) = \{x,y\}$. In this case, we see x chosen whenever the decision maker faces budget $\{x,y\}$, but we may see either x or y chosen when she faces budget $\{x,y,z\}$. ■

When using choice structures to model individual behavior, we may want to impose some "reasonable" restrictions regarding an individual's choice behavior. An important assumption, the weak axiom of revealed preference [first suggested by Samuelson; see Chapter 5 in Samuelson (6, 1947)], reflects the expectation that an individual's observed choices will display a certain amount of consistency. For example, if an individual chooses alternative x (and only that) when faced with a choice between x and y, we would be surprised to see her choose y when faced with a decision among x, y, and a third alterative z. The idea is that the choice of x when facing the alternatives $\{x,y\}$ reveals a proclivity for choosing x over y that we should expect to see reflected in the individual's behavior when faced with the alternatives $\{x,y,z\}$.

The weak axiom is stated formally in Definition 1.C.1.

Definition 1.C.1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property holds:

> If for some $B \in \mathscr{B}$ with $x,y \in B$ we have $x \in C(B)$, then for any B' in $\mathscr{B}$ with $x,y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

In words, the weak axiom says that if x is ever chosen when y is available, then there can be no budget set containing both alternatives for which y is chosen and x is not. Note how the assumption that choice behavior satisfies the weak axiom captures the consistency idea: If $C(\{x,y\}) = \{x\}$, then the weak axiom says that we cannot have $C(\{x,y,z\}) = \{y\}$.[4]

A somewhat simpler statement of the weak axiom can be obtained by defining a *revealed preference relation* $\succsim^*$ from the observed choice behavior in $C(\cdot)$.

[4] In fact, it says more: We must have $C(\{x,y,z\}) = \{x\}, = \{z\}$, or $= \{x,z\}$. You are asked to show this in Exercise 1.D. See also Exercise 1.D.

Definition 1.C.2. Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is defined by

$$x \succsim^* y \Leftrightarrow \text{ there is some } B \in \mathscr{B} \text{ such that } x, y \in B \text{ and } x \in C(B)$$

We read $x \succsim^* y$ as "x is revealed at least as good as y." Note that the revealed preference relation $\succsim^*$ need not be either complete or transitive. In particular, for any pair of alternatives x and y to be comparable, it is necessary that, for some $B \in \mathscr{B}$, we have $x, y \in B$ and either $x \in C(B)$ or $y \in C(B)$, or both.

We might also informally say that "x is revealed preferred to y" if there is some $B \in \mathscr{B}$ such that $x, y \in B$, $x \in C(B)$, and $y \notin C(B)$, that is, if x is ever chosen over y when both are feasible.

With this terminology, we can restate the weak axiom as follows: "If *x is revealed at least as good as y, then y cannot be revealed preferred to x*."

Example 1.C.2. Do the two choice structures considered in Example 1.C.1 satisfy the weak axiom? Consider choice structure $(\mathscr{B}, C_1(\cdot))$. With this choice structure, we have $x \succsim^* y$ and $x \succsim^* z$, but there is no revealed preference relationship that can be inferred between y and z. This choice structure satisfies the weak axiom because y and z are never chosen.

Now consider choice structure $(\mathscr{B}, C_2(\cdot))$. Because $C_2(\{x,y,z\}) = \{x,y\}$, we have $y \succsim^* x$ (as well as $x \succsim^* y$, $x \succsim^* z$, and $y \succsim^* z$). But because $C_2(\{x,y\}) = \{x\}$, x is revealed preferred to y. Therefore, the choice structure $(\mathscr{B}, C_2)$ violates the weak axiom. ■

We should note that the weak axiom is not the only assumption concerning choice behavior that we may want to impose in any particular setting. For example, in the consumer demand setting discussed in Chapter 2, we impose further conditions that arise naturally in that context.

The weak axiom restricts choice behavior in a manner that parallels the use of the rationality assumption for preference relations. This raises a question: What is the precise relationship between the two approaches? In Section 1.D, we explore this matter.

1.D The Relationship between Preference Relations and Choice Rules

We now address two fundamental questions regarding the relationship between the two approaches discussed so far:

(i) If a decision maker has a rational preference ordering $\succsim$, do her decisions when facing choices from budget sets in $\mathscr{B}$ necessarily generate a choice structure that satisfies the weak axiom?

(ii) If an individual's choice behavior for a family of budget sets $\mathscr{B}$ is captured by a choice structure $(\mathscr{B}, C(\cdot))$ satisfying the weak axiom, is there necessarily a rational preference relation that is consistent with these choices?

As we shall see, the answers to these two questions are, respectively, "yes" and "maybe".

To answer the first question, suppose that an individual has a rational preference relation $\succsim$ on X. If this individual faces a nonempty subset of alternatives $B \subset X$, her preference-maximizing behavior is to choose any one of the elements in the set:

$$C^*(B,\succsim) = \{x \in B : x \succsim y \text{ for every } y \in B$$

The elements of set $C^*(B,\succsim)$ are the decision maker's most preferred alternatives in B. In principle, we could have $C^*(B,\succsim) = \varnothing$ for some B; but if X is finite, or if suitable (continuity) conditions hold, then $C^*(B,\succsim)$ will be nonempty.[5] From now on, we will consider only preferences $\succsim$ and families of budget sets $\mathscr{B}$ such that $C^*(B,\succsim)$ is nonempty for all $B \in \mathscr{B}$. We say that the rational preference relation $\succsim$ generates the choice structure $(\mathscr{B}, C^*(\cdot,\succsim))$.

The result in Proposition 1.D.1 tells us that any choice structure generated by rational preferences necessarily satisfies the weak axiom.

Proposition 1.D.1. Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot,\succsim))$, satisfies the weak axiom.

Proof. Suppose that for some $B \in \mathscr{B}$, we have $x, y \in B$ and $x \in C^*(B,\succsim)$. By the definition of $C^*(B,\succsim)$, this implies $x \succsim y$. To check whether the weak axiom holds, suppose that for some $B' \in \mathscr{B}$ with $x, y \in B'$, we have $y \in C^*(B',\succsim)$. This implies that $y \succsim z$ for all $z \in B'$. But we already know that $x \succsim y$. Hence, by transitivity, $x \succsim z$ for all $z \in B'$, and so $x \in C^*(B',\succsim)$. This is precisely the conclusion that the weak axiom demands. *Q.E.D*

Proposition 1.D.1 constitutes the "yes" answer to our first question. That is, if behavior is generated by rational preferences then it satisfies the consistency requirements embodied in the weak axiom.

In the other direction (from choice to preferences), the relationship is more subtle. To answer this second question, it is useful to begin with a definition.

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B,\succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

In words, the rational preference relation $\succsim$ rationalizes choice rule $C(\cdot)$ on $\mathscr{B}$? if the optimal choices generated by $\succsim$ (captured by $C^*(\cdot,\succsim)$) coincide with $C(\cdot)$ for all budget sets in $\mathscr{B}$. In a sense, preferences explain behavior; we can interpret the decision maker's choices as if she were a preference maximizer. Note that in general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$ (see Exercise l.D.l).

Proposition 1.D.1 implies that the weak axiom must be satisfied if there is to be a rationalizing preference relation. In particular, since $C^*(\cdot,\succsim)$ satisfies the weak axiom for any $\succsim$, only a choice rule that satisfies the weak axiom can be rationalized. It turns out, however, that the weak axiom is not sufficient to ensure the existence. of a rationalizing preference relation.

[5] Exercise 1.D asks you to establish the nonemptiness of $C^*(B,\succsim)$ for the case where X is finite. For general results, See Section **??** of the Mathematical Appendix and Section 3.C for a specific application

Example 1.D.1. Suppose that $X = \{x,y,z\}$, $\mathscr{B} = \{\{x,y\},\{y,z\},\{x,z\}\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, and $C(\{x,z\}) = \{z\}$. This choice structure satisfies the weak axiom (you should verify this). Nevertheless, we cannot have rationalizing preferences. To see this, note that to rationalize the choices under $\{x,y\}$ and $\{y,z\}$ it would be necessary for us to have $x \succ y$ and $y \succ z$. But, by transitivity, we would then have $x \succ z$, which contradicts the choice behavior under $\{x,z\}$. Therefore, there can be no rationalizing preference relation. ■

To understand Example 1.D.1, note that the more budget sets there are in $\mathscr{B}$, the more the weak axiom restricts choice behavior; there are simply more opportunities for the decision maker's choices to contradict one another. In Example 1.D.1, the set $\{x,y,z\}$ is not an element of $\mathscr{B}$. As it happens, this is crucial (see Exercises 1.D). As we now show in Proposition 1.D.2, if the family of budget sets $\mathscr{B}$ includes enough subsets of X, and if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then there exists a rational preference relation that rationalizes $C(\cdot)$ relative to $\mathscr{B}$ [this was first shown by Arrow (1, 1959)].

Proposition 1.D.2. If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$ for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the only preference relation that does so.

Proof. The natural candidate for a rationalizing preference relation is the revealed preference relation $\succsim^*$. To prove the result, we must first show two things: (i) that $\succsim^*$ is a rational preference relation, and (ii) that $\succsim^*$ rationalizes $C(\cdot)$ on $\mathscr{B}$. We then argue, as point (iii), that $\succsim^*$ is the unique preference relation that does so.

(i) We first check that $\succsim^*$ is rational (i.e., that it satisfies completeness and transitivity).

Completeness By assumption (ii), $\{x,y\} \in \mathscr{B}$. Since either x or y must be an element of $C(\{x,y\})$, we must have $x \succsim^* y$, or $y \succsim^* x$, or both. Hence $\succsim^*$ is complete.

Transitivity Let $x \succsim^* y$ and $y \succsim^* z$. Consider the budget set $\{x,y,z\} \in \mathscr{B}$. It suffices to prove that $x \in C(\{x,y,z\})$, since this implies by the definition of $\succsim^*$ that $x \succsim^* z$. Because $C(\{x,y,z\}) \neq \varnothing$, at least one of the alternatives x, y, or z must be an element of $C(\{x,y,z\})$: Suppose that $y \in C(\{x,y,z\})$. Since $x \succsim^* y$, the weak axiom ,then yields $x \in C(\{x,y,z\})$, as we want. Suppose instead that $z \in C(\{x,y,z\})$; since $y \succsim^* z$, the weak axiom yields $y \in C(\{x,y,z\})$, and we are in the previous case.

(ii) We now show that $C(B) = C^*(B, \succsim^*)$ for all $B \in \mathscr{B}$; that is, the revealed preference relation $\succsim^*$ inferred from $C(\cdot)$ actually generates $C(\cdot)$. Intuitively, this seems sensible. Formally, we show this in two steps. First, suppose that $x \in C(B)$. Then $x \succsim^* y$ for all $y \in B$; so we have $x \in C^*(B, \succsim^*)$. This means that $C(B) \subset C^*(B, \succsim^*)$. Next, suppose that $x \in C^*(B, \succsim^*)$. This implies that $x \succsim^* y$ for all $y \in B$; and so for each $y \in B$, there must exist some set $B_y \in \mathscr{B}$ such that $x, y \in B_y$ and $x \in C(B_y)$. Because $C(B) \neq \varnothing$, the weak axiom then implies that $x \in C(B)$. Hence, $C^*(B, \succsim^*) \subset C(B)$. Together, these inclusion relations imply that $C(B) = C^*(B, \succsim^*)$.

(iii) To establish uniqueness, simply note that because $\mathscr{B}$ includes all two-element subsets of X, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over X of any rationalizing preference.

This completes the proof. *Q.E.D*

We can therefore conclude from Proposition 1.D.2 that for the special case in which choice is defined for all subsets of X, a theory based on choice satisfying the weak axiom is completely equivalent to a theory of decision making based on rational preferences. Unfortunately, this special case is too special for economics. For many situations of economic interest, such as the theory of consumer demand, choice is defined only for special kinds of budget sets. In these settings, the weak axiom does not exhaust the choice implications of rational preferences. We shall see in Section **??**, however, that a strengthening of the weak axiom (which imposes more restrictions on choice behavior) provides a necessary and sufficient condition for behavior to be capable of being rationalized by preferences.

Definition 1.D.1 defines a rationalizing preference as one for which $C(B) = C^*(B, \succsim)$. An alternative notion of a rationalizing preference that appears in the literature requires only that $C(B) \subset C^*(B, \succsim)$; that is, $\succsim$ is said to rationalize $C(\cdot)$ on $\mathscr{B}$ if $C(B)$ is a subset of the most preferred choices generated by $\succsim$, $C^*(B, \succsim)$, for every budget $B \in \mathscr{B}$.

There are two reasons for the possible use of this alternative notion. The first is, in a sense, philosophical. We might want to allow the decision maker to resolve her indifference in some specific manner, rather than insisting that indifference means that anything might be picked. The view embodied in Definition 1.D.1 (and implicitly in the weak axiom as well) is that if she chooses in a specific manner then she is, de facto, not indifferent.

The second reason is empirical. If we are trying to determine from data whether an individual's choice is compatible with rational preference maximization, we will in practice have only a finite number of observations on the choices made from any given budget set B. If $C(B)$ represents the set of choices made with this limited set of observations, then because these limited observations might not reveal all the decision maker's preference maximizing choices, $C(B) \subset C^*(B, \succsim)$ is the natural requirement to impose for a preference relationship to rationalize observed choice data.

Two points are worth noting about the effects of using this alternative notion. First, it is a weaker requirement. Whenever we can find a preference relation that rationalizes choice in the sense of Definition 1.D.1, we have found one that does so in this other sense, too. Second, in the abstract setting studied here, to find a rationalizing preference relation in this latter sense is actually trivial: Preferences that have the individual indifferent among all elements of X will rationalize any choice behavior in this sense. When this alternative notion is used in the economics literature, there is always an insistence that the rationalizing preference relation should satisfy some additional properties that are natural restrictions for the specific economic context being studied.

EXERCISES

Exercise 1.B.1 Prove property (iii) of Proposition 1.B.1.

Exercise 1.B.2 Prove properties (i) and (ii) of Proposition 1.B.1.

Exercise 1.B.3 Show that if $f:\mathbb{R}\to\mathbb{R}$ is a strictly increasing function and $u:X\to\mathbb{R}$ is a utility function representing preference relation $\succsim$, then the function $v:X\to\mathbb{R}$ defined by $v(x)=f(u(x))$ is also a utility function representing preference relation $\succsim$.

Exercise 1.B.4 Consider a rational preference relation $\succsim$. Show that if $u(x)=u(y)$ implies $x\sim y$,and if $u(x)>u(y)$ implies $x\succ y$, then $u(\cdot)$ is a utility function representing $\succsim$.

Exercise 1.B.5 Show that if X is finite and $\succsim$ is a rational preference relation on X, then there is a utility function $u:X\to\mathbb{R}$ that represents $\succsim$. [Hint: Consider first the case in which the individual's ranking between any two elements of X is strict (i.e., there is never any indifference), and construct a utility function representing these preferences; then extend your argument to the general case.]

Exercise 1.C.1 Consider the choice structure $(\mathscr{B},C(\cdot))$ with $\mathscr{R}=(\{x,y\},\{x,y,z\})$ and $C(\{x,y\})=\{x\}$. Show that if $(\mathscr{B},C(\cdot))$ satisfies the weak axiom, then we must have $C(\{x,y,z\})=\{x\},=\{z\}$, or $=\{x,z\}$.

Exercise 1.C.2 Show that the weak axiom (Definition 1.C.1) is equivalent to the following property holding:

> Suppose that $B,B'\in\mathscr{B}$, that $x,y\in B$, and that $x,y\in B'$. Then if $x\in C(B)$ and $y\in C(B')$, we must have $\{x,y\}\subset C(B)$ and $\{x,y\}\subset C(B')$.

Exercise 1.C.3 Suppose that choice structure $(\mathscr{B},C(\cdot))$ satisfies the weak axiom. Consider the following two possible revealed preferred relations, $\succsim^*$ and $\succsim^{**}$:

$x\succ^* y\Leftrightarrow$ there is some $B\in\mathscr{B}$ such that $x,y\in B$, $x\in C(B)$, and $y\notin C(B)$

$x\succsim^{**} y\Leftrightarrow x\succsim^* y$ but not $y\succsim^* x$

where $\succsim^*$ is the revealed at-least-as-good-as relation defined in Definition 1.C.2.

(a) Show that $\succsim^*$ and $\succsim^{**}$ give the same relation over X; that is, for any $x,y\in X$, $x\succsim^* y\Leftrightarrow x\succsim^{**} y$. Is this still true if $(\mathscr{B},C(\cdot))$ does not satisfy the weak axiom?

(b) Must $\succsim^*$ be transitive?

(c) Show that if $\mathscr{B}$ includes all three-element subsets of X, then $\succsim^*$ is transitive.

Exercise 1.D.1 Give an example of a choice structure that can be rationalized by several preference relations. Note that if the family of budgets $\mathscr{B}$ includes all the two-element subsets of X, then there can be at most one rationalizing preference relation.

Exercise 1.D.2 Show that if X is finite, then any rational preference relation generates a nonempty choice rule; that is, $C(B)\neq\varnothing$ for any $B\subset X$ with $B\neq\varnothing$.

Exercise 1.D.3 Let $X=\{x,y,z\}$, and consider the choice structure $(\mathscr{B},C(\cdot))$ with

$$\mathscr{B}=\{\{x,y\},\{y,z\},\{x,z\},\{x,y,z\}\}$$

and $C(\{x,y\})=\{x\}$, $C(\{y,z\})=\{y\}$, and $C(\{x,z\})=\{z\}$, as in Example 1.D.1. Show that $(\mathscr{B},C(\cdot))$ must violate the weak axiom.

Exercise 1.D.4 Show that a choice structure $(\mathscr{B},C(\cdot))$ for which a rationalizing preference relation $\succsim$ exists satisfies the *path-invariance* property: For every pair $B_1,B_2\in\mathscr{B}$ such that $B_1\cup B_2\in\mathscr{B}$ and $C(B_1)\cup C(B_2)\in\mathscr{B}$, we have $C(B_1\cup B_2)=C(C(B_1)\cup C(B_2))$, that is, the decision problem can safely be subdivided. See Plott (4, 1973) for further discussion.

Exercise 1.D.5 Let $X = \{x,y,z\}$ and $\mathscr{B} = \{\{x,y\},\{y,z\},\{z,x\}\}$. Suppose that choice is now stochastic in the sense that, for every $B \in \mathscr{B}$, $C(B)$ is a frequency distribution over alternatives in B. For example, if $B = \{x,y\}$, we write $C(B) = (C_x(B), C_y(B))$, where $C_x(B)$ and $C_y(B)$ are nonnegative numbers with $C_x(B) + C_y(B) = 1$. We say that the stochastic choice function $C(\cdot)$ can be *rationalized by preferences* if we can find a probability distribution Pr over the six possible (strict) preference relations on X such that for every $B \in \mathscr{B}$, $C(B)$ is precisely the frequency of choices induced by Pr. For example, if $B = \{x,y\}$, then ${}_x(B) = Pr(\{\succ : x \succ y\})$. This concept originates in Thurstone (8, 1927), and it is of considerable econometric interest (indeed, it provides a theory for the error term in observable choice).

(a) Show that the stochastic choice function $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\frac{1}{2}, \frac{1}{2})$ can be rationalized by preferences.

(b) Show that the stochastic choice function $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\frac{1}{4}, \frac{3}{4})$ is not rationalizable by preferences.

(c) Determine the $0 < \alpha < 1$ at which $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha, 1-\alpha)$ switches from rationalizable to nonrationalizable.

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Chapter 2
Consumer Choice

2.A Introduction

The most fundamental decision unit of microeconomic theory is the *consumer*. In this chapter, we begin our study of consumer demand in the context of a *market economy*. By a market economy, we mean a setting in which the goods and services that the consumer may acquire are available for purchase at known prices (or, equivalently, are available for trade for other goods at known rates of exchange).

We begin, in Sections 2.B to 2.D, by describing the basic elements of the consumer's decision problem. In Section 2.B, we introduce the concept of *commodities*, the objects of choice for the consumer. Then, in Sections 2.C and 2.D, we consider the physical and economic constraints that limit the consumer's choices. The former are captured in the *consumption set*, which we discuss in Section 2.C; the latter are incorporated in Section 2.D into the consumer's *Walrasian budget set*.

The consumer's decision subject to these constraints captured in the consumer's *Walrasian demand function*. In terms of the choice-based approach to individual decision making introduced in Section 1.C, the Walrasian demand function is the consumer's choice rule. We study this function and some of its basic properties in Section 2.E. Among them are what we call *comparative statics* properties: the ways in which consumer demand changes when economic constraints vary.

Finally, in Section 2.F, we consider the implications for the consumer's demand function of the *weak axiom of revealed preference*. The central conclusion we reach is that in the consumer demand setting, the weak axiom is essentially equivalent to the *compensated law of demand*, the postulate that prices and demanded quantities move in opposite directions for price changes that leave real wealth unchanged.

2.B Commodities

The decision problem faced by the consumer in a market economy is to choose consumption levels of the various goods and services that are available for purchase in the market. We call these goods and services *commodities*. For simplicity, we assume that the number of commodities is finite and equal to L (indexed by $= 1, \cdots, L$).

As a general matter, a *commodity vector* (or commodity *bundle*) is a list of amounts of the different commodities,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$

and can be viewed as a point in $\mathbb{R}^L$, the commodity *space*.[1]

We can use commodity vectors to represent an individual's consumption levels. The lth entry of the commodity vector stands for the amount of commodity l consumed. We then refer to the vector as a *consumption vector* or *consumption bundle*.

Note that time (or, for that matter, location) can be built into the definition of a commodity. Rigorously, bread today and tomorrow should be viewed as distinct commodities. In a similar vein, when we deal with decisions under uncertainty in Chapter 6, viewing bread in different "states of nature" as different commodities can be most helpful.

Although commodities consumed at different times should be viewed rigorously as distinct commodities, in practice, economic models often involve some "time aggregation." Thus, one commodity might be "bread consumed in the month of February" even though, in principle, bread consumed at each instant in February should be distinguished. A primary reason for stich time aggregation is that the economic data to which the model is being applied are aggregated in this way. The hope of the modeler is that the commodities being aggregated are sufficiently similar that little of economic interest is being lost.

We should also note that in some contexts it becomes convenient, and even necessary, to expand the set of commodities to include goods and services that may potentially be available for purchase but are not actually so and even some that may be available by means other than market exchange (say, the experience of "family togetherness"). For nearly all of what follows here, however, the narrow construction introduced in this section suffices.

2.C The Consumption Set

Consumption choices are typically limited by a number of physical constraints. The simplest example is when it may be impossible for the individual to consume a negative amount of a commodity such as bread or water.

Formally, the *consumption set* is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by his environment.

Consider the following four examples for the case in which $L = 2$:

(i) Figure 2.C.1 represents possible consumption levels of bread and leisure in a day. Both levels must be nonnegative and, in addition, the consumption of more than 24 hours of leisure in a day is impossible.

[1] Negative entries in commodity vectors will often represent debits or net outflows of goods. For example, in Chapter 5, the inputs of a firm are measured as negative numbers.

(ii) Figure 2.C.2 represents a situation in which the first good is perfectly divisible but the second is available only in' nonnegative integer amounts.

(iii) Figure 2.C.3 captures the fact that it is impossible to eat bread at the same instant in Washington and in New York. [This example is borrowed from Malinvaud (4, 1978).]

(iv) Figure 2.C.4 represents a situation where the consumer requires a minimum of four slices of bread a day to survive and there are two types of bread, brown and white.

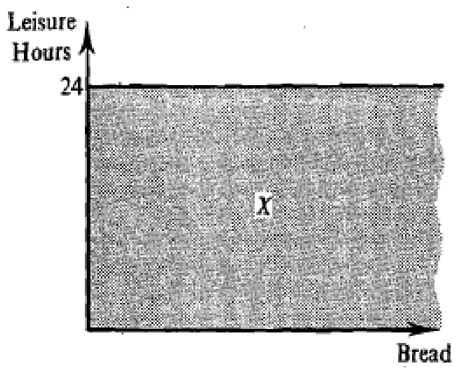

Figure 2.C.1 A consumption set.

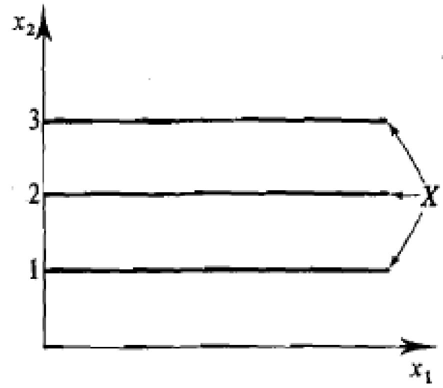

Figure 2.C.2 A consumption set where good 2 must be consumed in integer amounts.

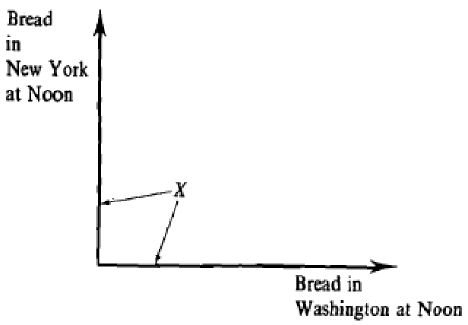

Figure 2.C.3 A consumption set where only one good can be consumed.

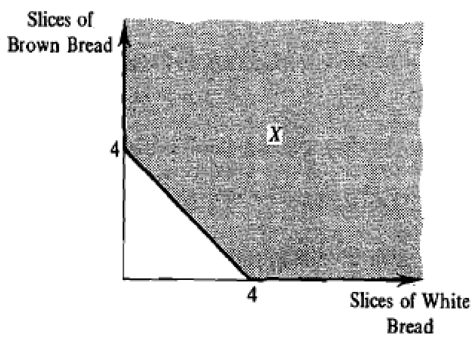

Figure 2.C.4 A consumption set reflecting survival needs.

In the four examples, the constraints are physical in a very literal sense. But the constraints that we incorporate into the consumption set can also be institutional in nature. For example, a law requiring that no one work more than 16 hours a day would change the consumption set in Figure 2.C.1 to that in Figure 2.C.5.

To keep things as straightforward as possible, we pursue our discussion adopting the simplest sort of consumption set:

$$X = \mathbb{R}^L_+ = \left\{x \in \mathbb{R}^L : x_l \geq 0 \text{ for } l = 1, \cdots, L\right\}$$

the set of all nonnegative bundles of commodities. It is represented in Figure 2.C.6. Whenever we consider any consumption set X other than $\mathbb{R}^L_+$, we shall be explicit about it.

One special feature of the set $\mathbb{R}^L_+$ is that it is *convex*. That is, if two consumption bundles x and x' are both elements of $\mathbb{R}^L_+$, then the bundle $x'' = \alpha x + (1-\alpha)x'$ is also an element of $\mathbb{R}^L_+$ for any $\alpha \in [0,1]$ (see Section **??** of the Mathematical Appendix for the definition and properties of convex sets).[2] The consumption sets in Figure 2.C.1, Figure 2.C.4, Figure 2.C.5, and Figure 2.C.6 are convex sets; those in Figure 2.C.2 and Figure 2.C.3 are not.

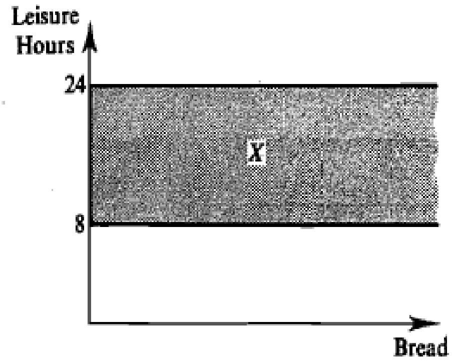

Figure 2.C.5 A consumption set reflecting a legal limit on the number of hours worked.

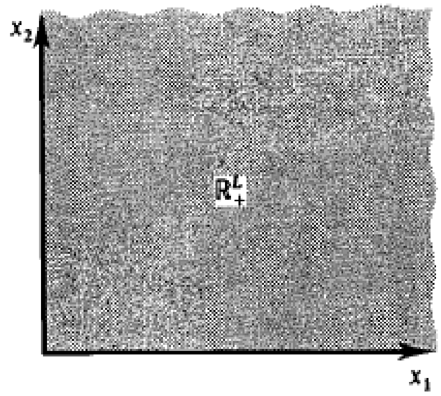

Figure 2.C.6 The consumption set $\mathbb{R}^L_+$

Much of the theory to be developed applies for general convex consumption sets as well as for $\mathbb{R}^L_+$. Some of the results, but not all, survive without the assumption of convexity.[3]

2.D Competitive Budgets

In addition to the physical constraints embodied in the consumption set, the consumer faces an important economic constraint: his consumption choice is limited to those commodity bundles that he can afford.

To formalize this constraint, we introduce two assumptions. First, we suppose that the L commodities are all traded in the market at dollar prices that are publicly quoted (this is the *principle of completeness*, or *universality*, *of markets*). Formally, these prices are represented by the *price vector*

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

which gives the dollar cost for a unit of each of the L commodities. Observe that there is nothing that logically requires prices to be positive. A negative price simply means that a “buyer” is actually paid to consume the commodity (which is not illogical for commodities that are “bads,” such as pollution). Nevertheless, for simplicity, here we always assume $p \gg 0$; that is, $p_l > 0$ for every l.

[2] Recall that $x'' = \alpha x + (1-\alpha)x'$ is a vector whose lth entry is $x''_l = \alpha x_l + (1-\alpha)x'_l$

[3] Note that commodity aggregation can help convexify the consumption set. In the example leading to 2.1Figure 2.C.3, the consumption set could reasonably be taken to be convex if the axes were instead measuring bread consumption over a period of a month.

Second, we assume that these prices are beyond the influence of the consumer. This is the so-called *price-taking assumption*. Loosely speaking, this assumption is likely to be valid when the consumer's demand for any commodity represents only a small fraction of the total demand for that good.

The affordability of a consumption bundle depends on two things: the market prices $p = (p_1, \cdots, p_L)$ and the consumer's wealth level (in dollars) w. The consumption bundle $x \in \mathbb{R}^L_+$ is affordable if its total cost does not exceed the consumer's wealth level w, that is, if[4]

$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq w$$

This economic-affordability constraint, when combined with the requirement that x lie in the consumption set $\mathbb{R}^L_+$, implies that the set of feasible consumption bundles consists of the elements of the set $\left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$. This set is known as the *Walrasian, or competitive budget set* (after Léon Walras).

Definition 2.D.1. The *Walrasian, or competitive budget set* $B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

The *consumer's problem*, given prices p and wealth w, can thus be stated as follows: *Choose a consumption bundle x from $B_{p,w}$.*

A Walrasian budget set $B_{p,w}$ is depicted in Figure 2.D.1 for the case of $L = 2$. To focus on the case in which the consumer has a nondegenerate choice problem, we always assume $w > 0$ (otherwise the consumer can afford only $x = 0$).

The set $\left\{x \in \mathbb{R}^L_+ : p \cdot x = w\right\}$) is called the *budget hyperplane* (for the case $L = 2$, we call it the *budget line*). It determines the upper boundary of the budget set. As Figure 2.D.1 indicates, the slope of the budget line when $L = 2$, $-(p_1/p_2)$, captures the rate of exchange between the two commodities. If the price of commodity 2 decreases (with p_1 and w held fixed), say to $\bar{p}_2 < p_2$, the budget set grows larger because more consumption bundles are affordable, and the budget line becomes steeper. This change is shown in Figure 2.D.2.

Another way to see how the budget hyperplane reflects the relative terms of exchange between commodities comes from examining its geometric relation to the price vector p. The price vector p, drawn starting from any point $\bar{x}$ on the budget hyperplane, must be orthogonal (perpendicular) to any vector starting at $\bar{x}$ and lying on the budget hyperplane, This is so because for any x' that itself lies on the budget hyperplane, we have $p \cdot x' = p \cdot \bar{x} = w$. Hence, $p \cdot \Delta x = 0$ for $\Delta x = (x' - \bar{x})$. Figure 2.D.3 depicts this geometric relationship for the case $L = 2$.[5]

[4] Often, this constraint is described in the literature as requiring that the cost of planned purchases not exceed the consumer's *income*. In either case, the idea is that the cost of purchases not exceed the consumer's available resources. We use the wealth terminology to emphasize that the consumer's actual problem may be intertemporal, with the commodities involving purchases over time, and the resource constraint being one of lifetime income (i.e., wealth) (see Exercise 2.F).

[5] To draw the vector p starting from $\bar{x}$, we draw a vector from point $(\bar{x}_1, \bar{x}_2)$ to point $(\bar{x}_1 + p_1, \bar{x}_2 + p_2)$. Thus, when we draw the price vector in this diagram, we use the "units" on the axes to represent units of prices rather than goods.

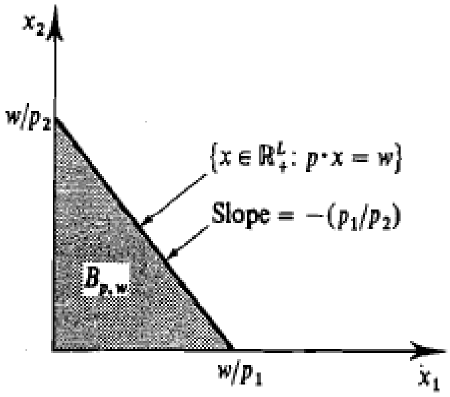

Figure 2.D.1 A Walrasian budget set.

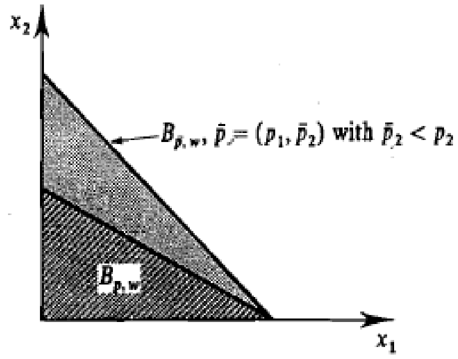

Figure 2.D.2 The effect of a price change on the Walrasian budget set.

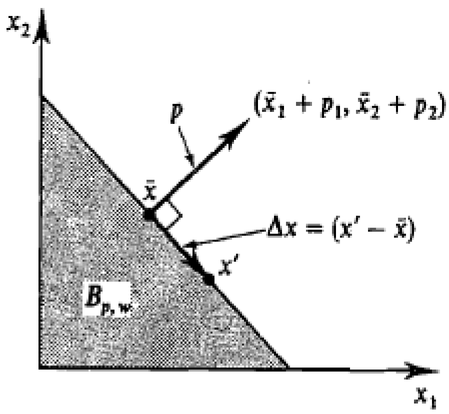

Figure 2.D.3 The geometric relationship between p and the budget hyperplane.

The Walrasian budget set $B_{p,w}$ is a *convex* set: That is, if bundles x and x' are both elements of $B_{p,w}$, then the bundle $x'' = \alpha x + (1-\alpha)x'$ is also. To see this, note first that because both x and x' are nonnegative, $x'' \in \mathbb{R}^L_+$. Second, since $p \cdot x \leq w$ and $p \cdot x' lew$, we have $p \cdot x'' = \alpha(p \cdot x) + (1-\alpha)(p \cdot x') lew$. Thus, $x'' \in B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$.

The convexity of $B_{p,w}$ plays a significant role in the development that follows. Note that the convexity of $B_{p,w}$ depends on the convexity of the consumption set $\mathbb{R}^L_+$. With a more general consumption set X, $B_{p,w}$ will be convex as long as X is. (See Exercise 2.F.)

Although Walrasian budget sets are of central theoretical interest, they are by no means the only type of budget set that a consumer might face in any actual situation. For example, a more realistic description of the market trade-off between a consumption good and leisure, involving taxes, subsidies, and several wage rates, is illustrated in Figure 2.D.4. In the figure, the price of the consumption good is 1, and the consumer earns wage rate s per hour for the first 8 hours of work and $s' > s$ for additional ("overtime") hours. He also faces a tax rate t per dollar on labor income earned above amount M. Note that the budget set in Figure 2.D.4 is not convex (you are asked to show this in Exercise 2.F). More complicated examples can readily be constructed and arise commonly in applied work. See Deaton and Muellbauer (2, 1980) and Burtless and Hausmann (1, 1975) for more illustrations of this sort.

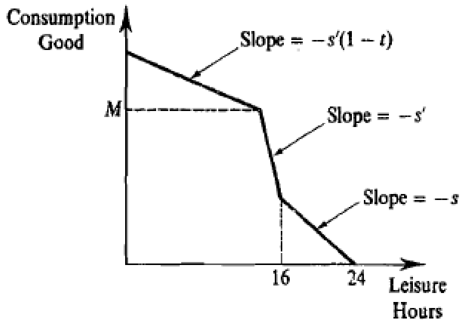

Figure 2.D.4 A more realistic description of the consumer's budget set.

2.E Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence* $x(p,w)$ assigns a set of chosen consumption bundles for each price-wealth pair (p,w). In principle, this correspondence can be multivalued; that is, there may be more than one possible consumption vector assigned for a given price-wealth pair (p,w). When this is so, any $x \in x(p,w)$ might be chosen by the consumer when he faces price-wealth pair (p,w). When $x(p,w)$ is single-valued, we refer to it as a *demand function*.

Throughout this chapter, we maintain two assumptions regarding the Walrasian demand correspondence $x(p,w)$: That it is *homogeneous of degree zero* and that it satisfies *Walras' law*.

Definition 2.E.1. The Walrasian demand correspondence $x(p,w)$ is *homogeneous of degree zero* if $x(\alpha p, \alpha w) = x(p,w)$ for any p, w and $\alpha > 0$.

Homogeneity of degree zero says that if both prices and wealth change in the same proportion, then the individual's consumption choice does not change. To understand this property, note that a change in prices and wealth from (p,w) to $(\alpha p, \alpha w)$ leads to no change in the consumer's set of feasible consumption bundles; that is, $B_{p,w} = B_{\alpha p, \alpha w}$. Homogeneity of degree zero says that the individual's choice depends only on the set of feasible points.

Definition 2.E.2. The Walrasian demand correspondence $x(p,w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p,w)$.

Walras' law says that the consumer fully expends his wealth. Intuitively, this is a reasonable assumption to make as long as there is some good that is clearly desirable. Walras' law should be understood, broadly: the consumer's budget may be an intertemporal one allowing for savings today to be used for purchases tomorrow. What Walras' law says is that the consumer fully expends his resources *over his lifetime*.

Exercise 2.E.1 Suppose $L = 3$, and consider the demand function $x(p,w)$ defined by

$$x_1 p, w = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1},$$

$$x_2 p, w = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2},$$

$$x_3 p, w = \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3},$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0,1)$?

In Chapter 3, where the consumer's demand $x(p,w)$ is derived from the maximization of preferences, these two properties (homogeneity of degree zero and satisfaction of Walras' law) hold under very general circumstances. In the rest of this chapter, however, we shall simply take them as assumptions about $x(p,w)$ and explore their consequences.

One convenient implication of $x(p,w)$ being homogeneous of degree zero can be noted immediately: Although $x(p,w)$ formally has $L+1$ arguments, we can, with no loss of generality, fix (*normalize*) the level of one of the $L+1$ independent variables at an arbitrary level. One common normalization is $p_l = 1$ for some l. Another is $w = 1$.[6] Hence, the effective number of arguments in $x(p,w)$ is L.

For the remainder of this section, we assume that $x(p,w)$ is always single-valued. In this case, we can write the function $x(p,w)$ in terms of commodity-specific demand functions:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_3(p,w) \end{bmatrix}$$

When convenient, we also assume $x(p,w)$ to be continuous and differentiable.

The approach we take here and in Section 2.F can be viewed as an application of the choice-based framework developed in Chapter 1. The family of Walrasian budget sets is $\mathscr{B}^{\mathscr{W}} = \{\mathscr{B}_{p,w} : p \gg 0, w > 0\}$. Moreover, by homogeneity of degree zero, $x(p,w)$ depends only on the budget set the consumer faces. Hence $(\mathscr{B}^{\mathscr{W}}, x(\cdot))$ is a choice structure, as defined in Section 1.C. Note that the choice structure $(\mathscr{B}^{\mathscr{W}}, x(cdot))$ does not include all possible subsets of X (e.g., it does not include all two- and three-element subsets of X). This fact will be significant for the relationship between the choice-based and preference-based approaches to consumer demand.

Comparative Statics

We are often interested in analyzing how the consumer's choice varies with changes in his wealth and in prices. The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

[6] We use normalizations extensively in Part **??**.

Wealth effects

For fixed prices $\bar{p}$, the function of wealth $x(\bar{p}, w)$ is called the consumer's *Engel function*. Its image in $\mathbb{R}^L_+, E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$, is known as the *wealth expansion path*. Figure 2.E.1 depicts such an expansion path.

At any (p, w), the derivative $\partial x_l(p, w)/\partial w$ is known as the *wealth effect* for the lth good.[7]

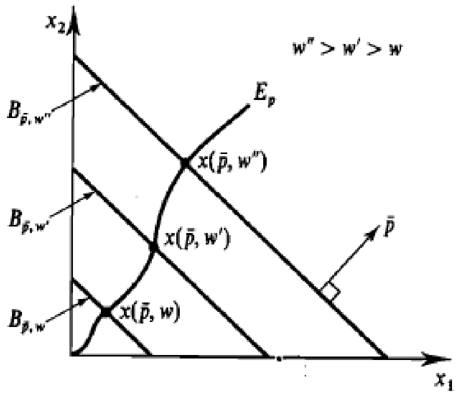

Figure 2.E.1 The wealth expansion path at prices $\bar{p}$.

A commodity l is *normal* at (p, w) if $\partial x_l(p, w)/\alpha w \geq 0$; that is, demand is nondecreasing in wealth. If commodity l's wealth effect is instead negative, then it is called *inferior* at (p, w). If every commodity is normal at all (p, w), then we say that demand is normal.

The assumption of normal demand makes sense if commodities are large aggregates (e.g., food, shelter). But if they are very disaggregated (e.g., particular kinds of shoes), then because of substitution to higher-quality goods as wealth increases, goods. that become inferior at some level of wealth may be the rule rather than the exception.

In matrix notation, the wealth effects are represented as follows:

$$D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \frac{\partial x_2(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_3(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L$$

Price effects

We can also ask how consumption levels of the various commodities change as prices vary.

Consider first the case where $L = 2$, and suppose we keep wealth and price p_1 fixed. Figure 2.E.2 represents the demand function for good 2 as a function of its own price p_2 for various levels of the price of good 1, with wealth held constant at amount w. Note that, as is customary in economics, the price variable, which here is the independent variable, is measured on the vertical axis, and the quantity demanded, the dependent variable, is measured on the horizontal axis. Another useful representation of the consumers' demand at different prices is the locus of

[7] It is also known as the *income effect* in the literature. Similarly, the wealth expansion path is sometimes referred to as an *income expansion path*.

points demanded in $\mathbb{R}^2_+$: as we range over all possible values of p_2. This is known as an *offer curve*. An example is presented in Figure 2.E.3.

More generally, the derivative $\partial x_l(p,w)/\partial p_k$ is known as the *price effect* of p_k, the price of good k, *on the demand for good* l. Although it may be natural to think that a fall in a good's price will lead the consumer to purchase more of it (as in Figure 2.E.3), the reverse situation is not an economic impossibility. Good l is said to be a *Giffen good* at (p,w) if $\partial x_l(p,w)/\partial p_l > 0$. For the offer curve depicted in Figure 2.E.4, good 2 is a Giffen good at $(\bar{p}, p_2', w)$.

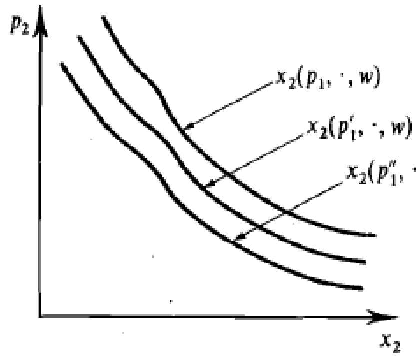

Figure 2.E.2 The demand for good 2 as a function of its price (for various levels of p_1).

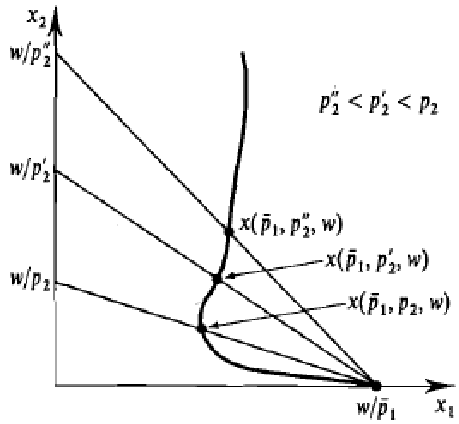

Figure 2.E.3 An offer curve.

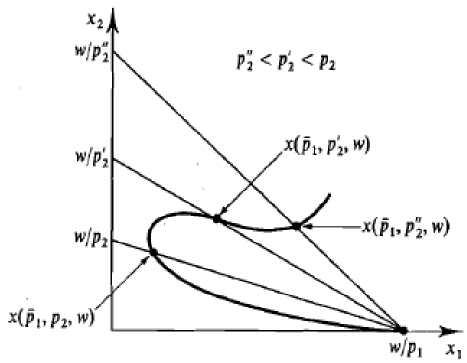

Figure 2.E.4 An offer curve where good 2 is inferior at $(\bar{p}, p_2', w)$.

Low-quality goods may well be Giffen goods for consumers with low wealth levels. For example, imagine that a poor consumer initially is fulfilling much of his dietary requirements with potatoes because they are a low-cost way to avoid hunger. If the price of potatoes falls, he can then afford to buy other, more desirable foods that also keep him from being hungry. His consumption of potatoes may well fall as a result. Note that the mechanism that leads to potatoes being a Giffen good in this story involves a wealth consideration: When the price of potatoes falls, the consumer is effectively wealthier (he can afford to purchase more generally), and so he buys fewer potatoes. We will be investigating this interplay between price and wealth effects more extensively in the rest of this chapter and in Chapter 3.

The price effects are conveniently represented in matrix form as follows:

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$$

Implications of homogeneity and Walras' law for price and wealth effects

Homogeneity and Walras' law imply certain restrictions on the comparative statics effects of consumer demand with respect to prices and wealth.

Consider, first, the implications of homogeneity of degree zero. We know that $x(\alpha p, \alpha w) - x(p,w) = 0$ for all $\alpha > 0$. Differentiating this expression with respect to α, and evaluating the derivative at $\alpha = 1$, we get the results shown in Proposition 2.E.1 (the result is also a special case of Euler's formula; see Section **??** of the Mathematical Appendix for details).

Proposition 2.E.1. If the Walrasian demand function $x(p,w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0 \text{ for } l = 1, \cdots, L. \tag{2.E.1}$$

In matrix notation, this is expressed as

$$D_p x(p,w)p + D_w x(p,w)w = 0. \tag{2.E.2}$$

Thus, homogeneity of degree zero implies that the price and wealth derivatives of demand for any good l, when weighted by these prices and wealth, sum to zero. Intuitively, this weighting arises because when we increase all prices and wealth proportionately, each of these variables changes in proportion to its initial level.

We can also restate equation (2.E.1) in terms of the *elasticities* of demand with respect to prices and wealth. These are defined, respectively, by

$$\varepsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l(p,w)}$$

and

$$\varepsilon_{lw}(p,w) = \frac{\partial x_l(p,w)}{\partial w} \frac{w}{x_l(p,w)}$$

These elasticities give the *percentage* change in demand for good l per (marginal) percentage change in the price of good k or wealth; note that the expression for $\varepsilon_{lw}(\cdot,\cdot)$ can be read as $(\Delta x/x)/(\Delta w/w)$. Elasticities arise very frequently in applied work. Unlike the derivatives of demand, elasticities are independent of the units chosen for measuring commodities and therefore provide a unit-free way of capturing demand responsiveness.

Using elasticities, condition (2.E.1) takes the following form:

$$\sum_{k=1}^{L} \varepsilon_{lk}(p,w) + \varepsilon_{lw}(p,w) = 0 \text{ for } l = 1, \cdots, L. \tag{2.E.3}$$

This formulation very directly expresses the comparative statics implication of homogeneity of degree zero: An equal percentage change in all prices and wealth leads to no change in demand.

Walras' law, on the other hand, has two implications for the price and wealth effects of demand. By Walras' law, we know that $p \cdot x(p,w) = w$ for all p and w. Differentiating this expression with respect to prices yields the first result, presented in Proposition 2.E.2.

Proposition 2.E.2. If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \text{ for } k = 1, \cdots, L, \tag{2.E.4}$$

or, written in matrix notion,[8]

$$\mathbf{p} \cdot \mathbf{D_p x(p,w)} + \mathbf{x(p,w)}^{\mathbf{T}} = \mathbf{0}^{\mathbf{T}} \tag{2.E.5}$$

Similarly, differentiating $p \cdot x(p,w) = w$ with respect to w, we get the second result, shown in Proposition 2.E.3.

Proposition 2.E.3. If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1, \tag{2.E.6}$$

or, written in matrix notation,

$$\mathbf{p} \cdot \mathbf{D_w x(p,w)} = \mathbf{1}. \tag{2.E.7}$$

The conditions derived in Propositions 2.E.2 and 2.E.3 are sometimes called the properties of *Cournot* and *Engel aggregation*, respectively. They are simply the differential versions of two facts: That total expenditure cannot change in response to a change in prices and that total expenditure must change by an amount equal to any wealth change.

Exercise 2.E.2 Show that equations (2.E.4) and (2.E.6) lead to the following two elasticity formulas:

$$\sum_{l=1}^{L} b_l(p,w)\varepsilon_{lk}(p,w) + b_k(p,w) = 0,$$

and

$$\sum_{l=1}^{L} b_l(p,w)\varepsilon_{lw}(p,w) = 1,$$

where $b_l(p,w) = p_l x_l(p,w)/w$ is the budget share of the consumer's expenditure on good l given prices p and wealth w.

2.F The Weak Axiom of Revealed Preference and the Law of Demand

In this section, we study the implications of the weak axiom of revealed preference for consumer demand. Throughout the analysis, we continue to assume that $x(p,w)$ is single-valued, homogeneous of degree zero, and satisfies Walras' law.[9]

[8] Recall that $\mathbf{0}^{\mathbf{T}}$ means a row vector of zeros.

[9] For generalizations to the case of multivalued choice, see Exercise 2.F.

The weak axiom was already introduced in Section 1.C as a consistency axiom for the choice-based approach to decision theory. In this section, we explore its implications for the demand behavior of a consumer. In the preference-based approach to consumer behavior to be studied in Chapter 3, demand necessarily satisfies the weak axiom. Thus, the results presented in Chapter 3, when compared with those in this section, will tell us how much more structure is imposed on consumer demand by the preference-based approach beyond what is implied by the weak axiom alone.[10]

In the context of Walrasian demand functions, the weak axiom takes the form stated in the Definition 2.F.1.

Definition 2.F.1. The Walrasian demand function $x(p,w)$ satisfies *the weak axiom of revealed preference* (the WA) if the following property holds for any two price-wealth situations (p,w) and (p',w'):

$$\text{if } p\cdot x(p',w') \leq w \text{ and } x(p',w') \neq x(p,w), \text{ then } p'\cdot x(p,w) > w'.$$

If you have already studied Chapter 1, you will recognize that this definition is precisely the specialization of the general statement of the weak axiom presented in Section 1.C to the context in which budget sets are Walrasian and $x(p,w)$ specifies a unique choice (see Exercise 2.F).

In the consumer demand setting, the idea behind the weak axiom can be put as follows: If $p\cdot x(p',w') \leq w$ and $x(p',w') \neq x(p,w)$, then we know that when facing prices p and wealth w, the consumer chose consumption bundle $x(p,w)$ even though bundle $x(p',w')$ was also affordable. We can interpret this choice as "revealing" a preference for $x(p,w)$ over $x(p',w')$. Now, we might reasonably expect the consumer to display some consistency in his demand behavior. In particular, given his revealed preference, we expect that he would choose $x(p,w)$ over $x(p',w')$ whenever they are both affordable. If so, bundle $x(p,w)$ must not be affordable at the price-wealth combination (p',w') at which the consumer chooses bundle $x(p',w')$. That is, as required by the weak axiom, we must have $p'\cdot x(p,w) > w'$.

The restriction on demand behavior imposed by the weak axiom when $L=2$ is illustrated in Figure 2.F.1. Each diagram shows two budget sets $B_{p',w'}$ and $B_{p'',w''}$ and their corresponding choice $x(p',w')$ and $x(p'',w'')$. The weak axiom tells us that we cannot have both $p'\cdot x(p'',w'') < w'$ and $p''\cdot x(p',w') \leq w''$. Panels (a) to (c) depict permissible situations, whereas demand in panels (d) and (e) violates the weak axiom.

Implications of the Weak Axiom

The weak axiom has significant implications for the effects of price changes on demand. We need to concentrate, however, on a special kind of price change.

As the discussion of Giffen goods in Section 2.E suggested, price changes affect the consumer in two ways. First, they alter the relative cost of different commodities. But, second, they

[10]Or, stated more properly, beyond what is implied by the weak axiom in conjunction with homogeneity of degree zero and Walras' law.

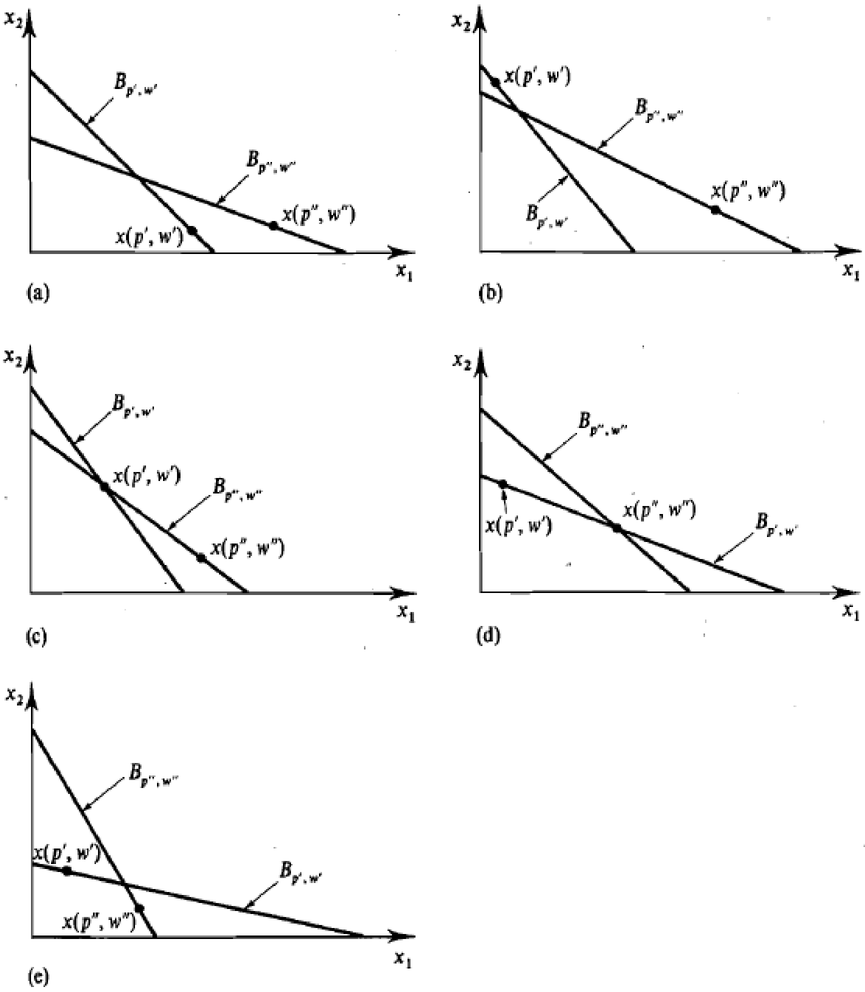

Figure 2.F.1 Demand in panels (a) to (c) satisfies the weak axiom; demand in panels (d) and (e) does not.

also change the consumer's real wealth: An increase in the price of a commodity impoverishes the consumers of that commodity. To study the implications of the weak axiom, we need to isolate the first effect.

One way to accomplish this is to imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes his initial consumption bundle just affordable at the new prices. That is, if the consumer is originally facing prices p and wealth w and chooses consumption bundle $x(p,w)$, then when prices change to p', we imagine that the consumer's wealth is adjusted to $w' = p' \cdot x(p,w)$. Thus, the wealth adjustment is $\Delta w = \Delta p \cdot x(p,w)$, where $\Delta p = (p' - p)$. This kind of wealth adjustment is known as *Slutsky wealth compensation*. Figure 2.F.2 shows the change in the budget set when a reduction in the price of good 1 from p_1 to p_1' is accompanied by Slutsky wealth compensation. Geometrically, the restriction is that the budget hyperplane corresponding to (p', w') goes through the vector $x(p,w)$.

We refer to price changes that are accompanied by such compensating wealth changes as

(Slutsky) compensated price changes.

In Proposition 2.F.1, we show that the weak axiom can be equivalently stated in terms of the demand response to compensated price changes.

Proposition 2.F.1. Suppose that the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p,w)$ satisfies the weak axiom if and only if the following property holds:
For any compensated price change from an initial situation (p,w) to a new price-wealth pair $(p',w') = (p', p'\cdot x(p,w))$, we have

$$(p'-p)\cdot\left[x(p',w')-x(p,w)\right] \leq 0, \tag{2.F.1}$$

with strict inequality whenever $x(p,w) \neq x(p',w')$.

Proof. (i) *The weak axiom implies inequality (2.F.1), with strict inequality if* $x(p,w) \neq x(p',w')$. The result is immediate if $x(p',w') = x(p,w)$, since then $(p'-p)\cdot[x(p',w')-x(p,w)] = 0$. So suppose that $x(p',w') \neq x(p,w)$. The left-hand side of inequality (2.F.1) can be written as

$$(p'-p)\cdot\left[x(p',w')-x(p,w)\right] = p'\cdot\left[x(p',w')-x(p,w)\right] - p\cdot\left[x(p',w')-x(p,w)\right]. \tag{2.F.2}$$

Consider the first term of (2.F.2). Because the change from p to p' is a compensated price change, we know that $p\cdot x(p,w) = w'$. In addition, Walras' law tells us that $w' = p'\cdot x(p',w')$. Hence

$$p'\cdot\left[x(p',w')-x(p,w)\right] = 0. \tag{2.F.3}$$

Now consider the second term of (2.F.2). Because $p'\cdot x(p,w) = w'$, $x(p,w)$ is affordable under price-wealth situation (p',w'). The weak axiom therefore implies that $x(p',w')$ must not be affordable under price-wealth situation (p,w). Thus, we must have $p\cdot x(p',w') > w$. Since $\cdot x(p,w) = w$ by Walras' law, this implies that

$$p\cdot\left[x(p',w')-x(p,w)\right] > 0 \tag{2.F.4}$$

Together, (2.F.2), (2.F.3) and (2.F.4) yield the result.

(ii) *The weak axiom is implied by (2.F.1) holding for all compensated price changes, with strict inequality if* $x(p,w) \neq x(p',w')$. The argument for this direction of the proof uses the following fact: The weak axiom holds if and only if it holds for all compensated price changes. That is, the weak axiom holds if, for any two price-wealth pairs (p,w) and (p',w'), we have $p' - x(p,w) > w'$ whenever $p\cdot x(p',w') = w$ and $x(p',w') \neq x(p,w)$.

To prove the fact stated in the preceding paragraph, we argue that if the weak axiom is violated, then there must be a compensated price change for which it is violated. To see this, suppose that we have a violation of the weak axiom, that is, two price-wealth pairs (p',w') and (p'',w'') such that $x(p',w') \neq x(p'',w'')$, $p'\cdot x(p'',w'') \leq w'$, and $p''\cdot x(p',w') \leq w''$. If one of these two weak inequalities holds with equality, then this is actually a compensated price change and we are done. So assume that, as shown in Figure 2.F.3, we have $p'\cdot x(p'',w'') < w'$ and $p''\cdot x(p',w') < w''$.

Now choose the value of a $\alpha \in (0,1)$ for which

$$(\alpha p' + (1-\alpha)p'')\cdot x(p',w') = (\alpha p' + (1-\alpha)p'')\cdot x(p'',w''),$$

and denote $p = \alpha p' + (1-\alpha)p''$ and $w = (\alpha p' + (1-\alpha)p'')\cdot x(p',w')$. This construction is illustrated in Figure 2.F.3. We then have

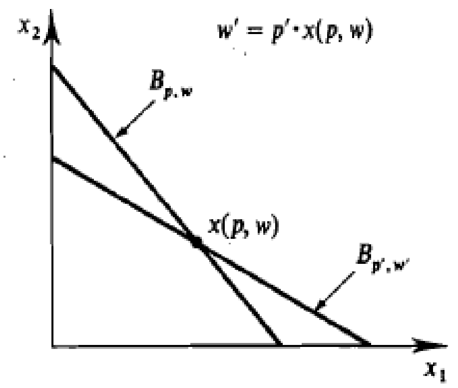

Figure 2.F.2 A compensated price change from (p,w) to (p',w').

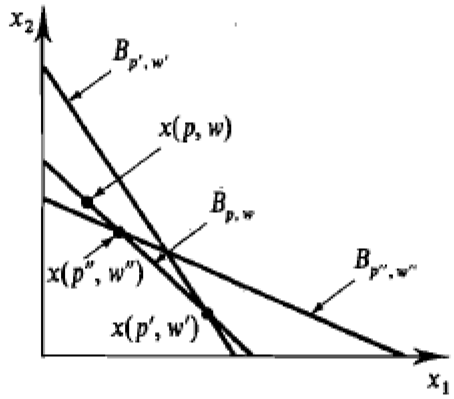

Figure 2.F.3 A compensated price change from (p,w) to (p',w').

$$\begin{aligned}
\alpha w' + (1-\alpha) w'' &> \alpha p' \cdot x(p',w') + (1-\alpha) p'' \cdot x(p',w') \\
&= w \\
&= p \cdot x(p,w) \\
&= \alpha p' \cdot x(p,w) + (1-\alpha) p'' \cdot x(p,w)
\end{aligned}$$

Therefore, either $p' \cdot x(p',w') < w'$ or $p'' \cdot x(p,w) < w''$. Suppose that the first possibility holds (the argument is identical if it is the second that holds). Then we have $x(p,w) \neq x(p',w')$, $p \cdot x(p',w') = w$, and $p' \cdot x(p,w) < w'$, which constitutes a violation of the weak axiom for the compensated price change from (p',w') to (p,w).

Once we know that in order to test for the weak axiom it suffices to consider only compensated price changes, the remaining reasoning is straightforward. If the weak axiom does not hold, there exists a compensated price change from some (p',w') to some (p,w) such that $x(p,w) \neq x(p',w')$, $p \cdot x(p',w') = w$, and $p' \cdot x(p',w') \leq w'$. But since $x(\cdot,\cdot)$ satisfies Walras' law, these two inequalities imply

$$p \cdot \left[x(p',w') - x(p,w)\right] = 0 \text{ and } p' \cdot \left[x(p',w') - x(p,w)\right] \geq 0$$

Hence, we would have

$$(p'-p) \cdot \left[x(p',w') - x(p,w)\right] \geq 0 \text{ and } x(p,w) \neq x(p',w'),$$

which is a contradiction to (2.F.1) holding for all compensated price changes [and with strict inequality when $x(p,w) \neq x(p',w')$]. *Q.E.D*

The inequality (2.F.1) can be written in shorthand as $\Delta p \cdot \Delta x \leq 0$, where $\Delta p = (p'-p)$ and $\Delta x = [x(p',w') - x(p,w)]$. It can be interpreted as a form of the *law of demand: Demand and price move in opposite directions*. Proposition 2.F.1 tells us that the law of demand holds for *compensated* price changes. We therefore call it the *compensated law of demand*.

The simplest case involves the effect on demand for some good l of a compensated change in its own price p_l. When only this price changes, we have $\Delta p = (0, \cdots, 0, \Delta p_l, 0, \cdots, 0)$. Since $\Delta p \cdot \Delta x = \Delta p_l \Delta x_l$,, Proposition 2.F.1 tells us that if $\Delta p_l > 0$, then we must have $\Delta x_l < 0$. The basic argument is illustrated in Figure 2.F.4. Starting at (p,w), a compensated decrease in the price of good 1 rotates the budget line through $x(p,w)$. The WA allows moves of demand only in the direction that increases the demand of good 1.

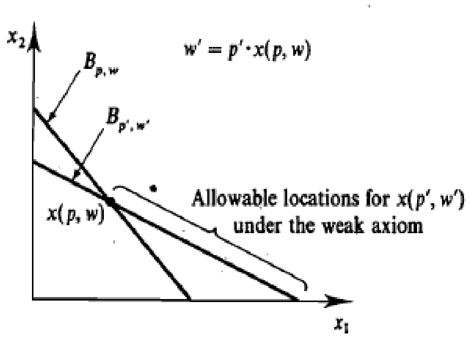

Figure 2.F.4 Demand must be nonincreasing in own price for a compensated price change.

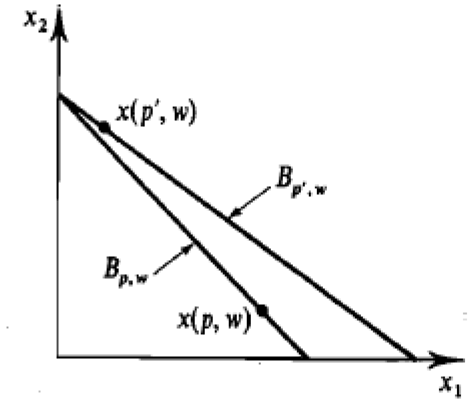

Figure 2.F.5 Demand for good 1 can fall when its price decreases for an uncompensated price change.

Figure 2.F.5 should persuade you that the WA (or, for that matter, the preference maximization assumption discussed in Chapter 3) is not sufficient to yield the law of demand for price changes that are *not* compensated. In the figure, the price change from p to p' is obtained by a decrease in the price of good 1, but the weak axiom imposes no restriction on where we place the new consumption bundle; as drawn, the demand for good 1 falls.

When consumer demand $x(p,w)$ is a differentiable function of prices and wealth, Proposition 2.F.1 has a differential implication that is of great importance. Consider, starting at a given price-wealth pair (p,w), a differential change in prices dp. Imagine that we make this a compensated price change by giving the consumer compensation of $dw = x(p,w) \cdot dp$ [this is just the differential analog of $\Delta w = x(p,w) \cdot \Delta p$]. Proposition 2.F.1 tells us that

$$dp \cdot dx \leq 0 \tag{2.F.5}$$

Now, using the chain rule, the differential change in demand induced by this compensated price change can be written as.

$$dx = D_p x(p,w)dp + D_w x(p,w)dw \tag{2.F.6}$$

Hence

$$dx = D_p x(p,w)dp + D_w x(p,w)\left[x(p,w) \cdot dp\right] \tag{2.F.7}$$

or equivalently

$$dx = \left[D_p x(p,w) + D_w x(p,w)x(p,w)^T\right] dp \tag{2.F.8}$$

Finally, substituting (2.F.8) into (2.F.5) we conclude that for any possible differential price change dp, we have

$$dp \cdot \left[D_p x(p,w) + D_w x(p,w)x(p,w)^T\right] dp \leq 0 \tag{2.F.9}$$

The expression in square brackets in condition (2.F.9) is an LxL matrix, which we denote by $S(p,w)$. Formally

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix},$$

where the (l,k)th entry is

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \tag{2.F.10}$$

The matrix $S(p,w)$ is known as the *substitution*, or *Slutsky*, matrix, and its elements are known as *substitution effects*.

The "substitution" terminology is apt because the term $s_{lk}(p,w)$ measures the differential change in the.consumption of commodity l (i.e., the substitution to or from other commodities) due to a differential change in the price of commodity k when wealth is adjusted so that the consumer can still just afford his original consumption bundle (i.e., due solely to a change in relative prices). To see this, note that the change in demand for good l if wealth is left unchanged is $(\partial x_l(p,w)/\partial p_k)dp_k$. For the consumer to be able to "just afford" his original consumption bundle, his wealth must vary by the amount $x_k(p,w)dp_k$. The effect of this wealth change on the demand for good l is then $(\partial x_l(p,w)/\partial w)[x_k(p,w)dp_k]$. The sum of these two effects is therefore exactly $_{lk}(p,w)dp_k$.

We summarize the derivation in equations (2.F.5) to (2.F.10) in Proposition 2.F.2.

Proposition 2.F.2. If a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p,w), the Slutsky matrix $S(p,w)$ satisfies $v \cdot S(p,w) \cdot v \leq 0$ for any $v \in \mathbb{R}^L$.

A matrix satisfying the property in Proposition 2.F.2 is called *negative semidefinite* (it is negative *definite* if the inequality is strict for all $v \neq 0$). See Section **??** of the Mathematical Appendix for more on these matrices.

Note that $S(p,w)$ being negative semidefinite implies that $s_{ll}(p,w) \leq 0$: That is, *the substitution effect of good l with respect to its own price is always nonpositive*.

An interesting implication of $s_{ll}(p,w) \leq 0$ is that a good can be a Giffen good at (p,w) only if it is inferior. In particular, since

$$s_{ll}(p,w) = \partial x_l(p,w)/\partial p_l + [\partial x_l(p,w)/\partial w]\, x_l(p,w) \leq 0,$$

if $\partial x_l(p,w)/\partial p_l > 0$, we must have $\partial x_l(p,w)/\partial w < 0$.

For later reference, we note that Proposition 2.F.2 does not imply, in general, that the matrix $S(p,w)$ is symmetric.[11] For $L = 2$, $S(p,w)$ is necessarily symmetric (you are asked to show this in Exercise 2.F). When $L > 2$, however, $S(p,w)$ need not be symmetric under the assumptions made so far (homogeneity of degree zero, Walras' law, and the weak axiom). See Exercises

[11] A matter of terminology: It is commsn in the mathematical literature that "definite" matrices are assumed to be symmetric. Rigorously speaking, if no symmetry is implied, the matrix would be called "quasidefinite." To simplify terminology, we use "definite" without any supposition about symmetry; if a matrix is symmetric, we say so explicitly. (See Exercise 2.F.)

2.F and 2.F for examples. In Chapter 3 (Section **??**), we shall see that the symmetry of $S(p,w)$ is intimately connected with the possibility of generating demand from the maximization of rational preferences.

Exploiting further the properties of homogeneity of degree zero and Walras' law, we can say a bit more about the substitution matrix $S(p,w)$.

Proposition 2.F.3. Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p,w) = 0$ and $S(p,w)p = 0$ for any (p,w).

Exercise 2.F.7 Prove Proposition 2.F.3. [*Hint*: Use Propositions 2.E.1 to 2.E.3.]

It follows from Proposition 2.F.3 that the matrix $S(p,w)$ is always singular (i.e., it has rank less, than L), and so the negative semidefiniteness of $S(p,w)$ established in Proposition 2.F.2 cannot be extended to negative definiteness (e.g., see Exercise 2.F).

Proposition 2.F.2 establishes negative semidefiniteness of $S(p,w)$ as a necessary implication of the weak axiom. One might wonder: Is this property sufficient to imply the WA [so that negative semidefiniteness of $S(p,w)$ is actually equivalent to the WA]? That is, if we have a demand function $x(p,w)$ that satisfies Walras' law, homogeneity of degree zero and has a negative semidefinite substitution matrix, must it satisfy the weak axiom? The answer is *almost, but not quite*. Exercise 2.F provides an example of a demand function with a negative semidefinite substitution matrix that violates the WA. The sufficient condition is that $v \cdot S(p,w)v < 0$ whenever $v \neq \alpha p$ for any scalar α; that is, $S(p,w)$ must be negative definite for all vectors other than those that are proportional to p. This result is due to Samuelson [see Samuelson (6, 1947) or Kihlstrom, Mas-Colell, and Sonnenschein (5, 1976) for an advanced treatment]. The gap between the necessary and sufficient conditions is of the same nature as the gap between the necessary and the sufficient second-order conditions for the minimization of a function.

Finally, how would a theory of consumer demand that is based solely on the assumptions of homogeneity of degree zero, Walras' law, and the consistency requirement embodied in the weak action compare with one based on rational preference maximization?

Based on Chapter 1, you might hope that Proposition 1.D.2 implies that the two are equivalent. But we cannot appeal to that proposition here because the family of Walrasian budgets does not include every possible budget; in particular, it does not include all the budgets formed by only two- or three-commodity bundles.

In fact, the two theories are not equivalent. For Walrasian demand and functions, the theory derived from the weak axiom is weaker than the theory derived from rational preferences, in the sense of implying fewer restrictions. This is shown formally in Chapter 3, where we demonstrate that if demand is generated from preferences, or is capable of being so generated, then it must have a symmetric Slutsky matrix at all (p,w). But for the moment, Example 2.F.1, due originally to Hicks (3, 1956), may be persuasive enough.

Example 2.F.1. In a three-commodity world, consider the three budget sets determined by the price vectors $p^1 = (2,1,2)$, $p^2 = (2,2,1)$, $p^3 = (1,2,2)$ and wealth $= 8$ (the same for the three budgets). Suppose that the respective (unique) choices are $x^1 = (1,2,2)$, $x^2 = (2,1,2)$, $x^3 = (2,2,1)$. In Exercise 2.F, you are asked to verify that any two pairs of choices satisfy the WA but that x^3 is revealed preferred to x^2, x^2 is revealed preferred to x^l, and x^1 is revealed preferred to x^3. This situation is incompatible with the existence of underlying rational preferences (transitivity would be violated). ■

The reason this example is only *persuasive* and does not quite settle. the question is that demand has been defined only for the three given budgets, therefore, we cannot be sure that it satisfies the requirements of the WA for all possible competitive budgets. To clinch the matter we refer to Chapter 3.

In summary, there are three primary conclusions to be drawn from Section 2.F:

(i) The consistency requirement embodied in the weak axiom (combined with the homogeneity of degree zero and Walras' law) is equivalent to the compensated law of demand.

(ii) The compensated law of demand, in turn, implies negative semidefiniteness of the substitution matrix $S(p,w)$.

(iii) These assumptions do not imply symmetry of $S(p,w)$, except in the case where $L = 2$.

EXERCISES

Exercise 2.D.1 A consumer lives for two periods, denoted 1 and 2, and consumes a single consumption good in each period. His wealth when born is $w > 0$. What is his (lifetime) Walrasian budget set?

Exercise 2.D.2 A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumer's Walrasian budget set?

Exercise 2.D.3 Consider an extension of the Walrasian budget set to an arbitrary consumption set $X : B_{p,w} = \{x \in X : p \cdot x \leq w\}$. Assume $(p,w) \gg 0$.

(a) If X is the set depicted in Figure 2.C.3, would $B_{p,w}$ be convex?

(b) Show that if X is a convex set, then $B_{p,w}$ is as well.

Exercise 2.D.4 Show that the budget set in Figure 2.D.4 is not convex.

Exercise 2.E.1 In text.

Exercise 2.E.2 In text.

Exercise 2.E.3 Use Propositions 2.E.1 to 2.E.3 to show that $p \cdot D_p x(p,w)p = -w$. Interpret.

Exercise 2.E.4 Show that if $x(p,w)$ is homogeneous of degree one with respect to w [ie., $x(p,\alpha w) = \alpha x(p,w)$ for all $\alpha > 0$] and satisfies Walras' law, then $\varepsilon_{lw}(p,w) = 1$ for every l. Interpret. Can you say something about $D_w x(p,w)$ and the form of the Engel functions and curves in this case?

Exercise 2.E.5 Suppose that $x(p,w)$ is a demand function which is homogeneous of degree one with respect to w and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is $\partial x_l(p,w)/\partial p_k = 0$ whenever $k \neq l$. Show that this implies that for every $l, x_l(p,w) = \alpha w/p_l$, where $\alpha_l > 0$ is a constant independent of (p,w).
Exercise 2.E.6 Verify that the conclusions of Propositions 2.E.1 to 2.E.3 hold for the demand function given in Exercise 2.E when $\beta = 1$.
Exercise 2.E.7 A consumer in a two-good economy has a demand function $x(p,w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p,w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?
Exercise 2.E.8 Show that the elasticity of demand for good l with respect to price $p_k, \varepsilon_{lk}(p,w)$, can be written as $\varepsilon_{lk}(p,w) = d\ln(x_l(p,w))/d\ln(p_k)$, where $\ln(\cdot)$ is the natural logarithm function. Derive a similar expression for $\varepsilon_{lw}(p,w)$. Conclude that if we estimate the parameters $(\alpha_0, \alpha_1, \alpha_2, \gamma)$ of the equation $\ln(x_l(p,w)) = \alpha_0, \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$, these parameter estimates provide us with estimates of the elasticities $\varepsilon_{l1}(p,w)$, $\varepsilon_{l2}(p,w)$, and $\varepsilon_{lw}(p,w)$.
Exercise 2.F.1 Show that for Walrasian demand functions, the definition of the weak axiom given in Definition 2.F.1 coincides with that in Definition 1.C.1.
Exercise 2.F.2 Verify the claim of Example 2.F.1.
Exercise 2.F.3 You are given the following partial information about a consumer's purchases. He consumes only two goods.

	Year1		Year2	
	Quantity	Price	Quantity	Price
Good 1	100	100	120	100
Good 2	100	100	?	80

Over what range of quantities of good 2 consumed in year 2 would you conclude:
(a) That his behaviour is inconsistent (i.e., in contradiction with the weak axiom)?
(b) That the consumer's consumption bundle in year 1 is revealed preferred to that in year 2?
(c) That the consumer's consumption bundle in year 2 is revealed preferred to that in year l?
(d) That there is insufficient information to justify (a), (b), and/or (c)?
(e) That good 1 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.
(f) That good 2 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.

Exercise 2.F.4 Consider the consumption of a consumer in two different periods, period 0 and period 1. Period t prices, wealth, and consumption are p^t, w_t, and $x^t = x(p^t, w_t,)$, respectively. It is often of applied interest to form an index measure of the quantity consumed by a consumer. The *Laspeyres* quantity index computes the change in quantity using period 0 prices as weights: $L_Q = (p^0 \cdot x^1)/(p^0 \cdot x^0)$. The *Paasche* quantity index instead uses period 1 prices as weights: $P_Q = (p^1 \cdot x^1)/(p^1 \cdot x^0)$. Finally, we could use the consumer's expenditure change: $E_Q = (p^1 \cdot x^1)/(p^0 \cdot x^0)$. Show the following:

(a) If $L_Q < 1$, then the consumer has a revealed preference for x^0 over x^1.

(b) If $P_Q > 1$, then the consumer has a revealed preference for x^1 over x^0.

(c) No revealed preference relationship is implied by either $E_Q > 1$ or $E_Q < 1$. Note that at the aggregate level, E_Q corresponds to the percentage change in gross national product.

Exercise 2.F.5 Suppose that $x(p,w)$ is a differentiable demand function that satisfies the weak axiom, Walras' law, and homogeneity of degree zero. Show that if $x(\cdot,\cdot)$ is homogeneous of degree one with respect to w [i.e., $x(p,\alpha w) = \alpha x(p,w)$ for all (p,w) and $\alpha > 0$], then the law of demand holds even for uncompensated price changes. If this is easier, establish only the infinitesimal version of this conclusion; that is, $dp \cdot D_p x(p,w) dp \leq 0$ for any dp.

Exercise 2.F.6 Suppose that $x(p,w)$ is homogeneous of degree zero. Show that the weak axiom holds if and only if for some $w > 0$ and all p, p' we have $p' \cdot x(p,w) > w$ whenever $p \cdot x(p',w) \leq w$ and $x(p',w) \neq x(p,w)$.

Exercise 2.F.7 In text.

Exercise 2.F.8 Let $\hat{s}_{lk}(p,w) = [p_k/x_l(p,w)]\, s_{li}(p,w)$ be the substitution terms in elasticity form. Express $\hat{s}_{lk}(p,w)$ in terms of $\varepsilon_{lk}(p,w)$, $\varepsilon_{lw}(p,w)$, and $b_k(p,w)$.

Exercise 2.F.9 A symmetric $n \times n$ matrix A is negative definite if and only if $(-1)^k |A_{kk}| > 0$ for all $k \leq n$, where A_{kk}, is the submatrix of A obtained by deleting the last $n-k$ rows and columns. For semidefiniteness of the symmetric matrix A, we replace the strict inequalities by weak inequalities and require that the weak inequalities hold for all matrices formed by permuting the rows and columns of A (see Section M.D of the Mathematical Appendix for details).

(a) Show that an arbitrary (possibly nonsymmetric) matrix A is negative definite (or semidefinite) if and only if $A + A^T$ is negative definite (or semidefinite). Show also that the above determinant condition (which can be shown to be necessary) is no longer sufficient in the nonsymmetric case.

(b) Show that for $L = 2$, the necessary and sufficient condition for the substitution matrix $S(p,w)$ of rank 1 to be negative semidefinite is that any diagonal entry (i.e., any own-price substitution effect) be negative.

Exercise 2.F.10 Consider the demand function in Exercise 2.E with $\beta = 1$. Assume that $w = 1$.

(a) Compute the substitution matrix. Show that at $p = (1,1,1)$, it is negative semidefinite but not symmetric.

(b) Show that this demand function does not satisfy the weak axiom. [*Hint*: Consider the price vector $p = (1,1,\varepsilon)$ and show that the substitution matrix is not negative semidefinite (for $\varepsilon > 0$ small).]

Exercise 2.F.11 Show that for $L = 2$, $S(p,w)$ is always symmetric. [*Hint*: Use Proposition 2.F.3.]

Exercise 2.F.12 Show that if the Walrasian demand function $x(p,w)$ is generated by a rational preference relation, than it must satisfy the weak axiom.

Exercise 2.F.13 Suppose that $x(p,w)$ may be multivalued.

(a) From the definition of the weak axiom given in Section 1.C, develop the generalization of Definition 2.F.1 for Walrasian demand correspondences.

(b) Show that if $x(p,w)$ satisfies this generalization of the weak axiom and Walras' law, then $x(\cdot)$ satisfies the following property:

$$\text{(*) For any } x \in x(p,w) \text{ and } x' \in x(p',w'), \text{ if } p \cdot x' < w, \text{ then } p \cdot x > w.$$

(c) Show that the generalized weak axiom and Walras' law implies the following generalized version of the compensated law of demand: Starting from any initial position (p,w) with demand $x \in x(p,w)$, for any compensated price change to new prices p' and wealth level $w' = p' \cdot x$, we have

$$(p' - p) \cdot (x' - x) \leq 0$$

for all $x' \in x(p',w')$, with strict inequality if $x' \in x(p,w)$.

(d) Show that if $x(p,w)$ satisfies Walras' law and the generalized compensated law of demand defined in (c), then $x(p,w)$ satisfies the generalized weak axiom.

Exercise 2.F.14 Show that if $x(p,w)$ is a Walrasian demand function that satisfies the weak axiom, then $x(p,w)$ must be homogeneous of degree zero.

Exercise 2.F.15 Consider a setting with $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. The consumer's demand function $x(p,w)$ satisfies homogeneity of degree zero, Walras' law and (fixing $p_3 = 1$) has

$$x_1(p,w) = -p_1 + p_2$$

and

$$x_2(p,w) = -p_2$$

Show that this demand function satisfies the weak axiom by demonstrating that its substitution matrix satisfies $v \cdot S(p,w)v < 0$ for all $v \neq \alpha p$. [*Hint*: Use the matrix results recorded in Section M.D of the Mathematical Appendix.] Observe then that the substitution matrix is not symmetric. (*Note*: The fact that we allow for negative consumption levels here is not essential for finding a demand function that satisfies the weak axiom but whose substitution matrix is not symmetric; with a consumption set allowing only for nonnegative consumption levels, however, we would need to specify a more complicated demand function.)

Exercise 2.F.16 Consider a setting where $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. Suppose that his demand function $x(p,w)$ is

$$x_1(p,w) = \frac{p_2}{p_3},$$

$$x_2(p,w) = -\frac{p_1}{p_3},$$

$$x_3(p,w) = \frac{w}{p_3}.$$

(a) Show that $x(p,w)$ is homogeneous of degree zero in (p,w) and satisfies Walras' law.

(b) Show that $x(p,w)$ violates the weak axiom.

(c) Show that $v \cdot S(p,w)v = 0$ for all $v \in \mathbb{R}^3$.

Exercise 2.F.17 In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p,w) = \frac{w}{\left(\sum_{l=1}^{L} p_l\right)} \text{ for } k = 1, \cdots, L.$$

(a) Is this demand function homogeneous of degree zero in (p,w)?

(b) Does it satisfy Walras' law?

(c) Does it satisfy the weak axiom?

(d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

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Chapter 3
Classical Demand Theory

3.A Introduction

In this chapter, we study the classical, preference-based approach to consumer demand.

We begin in Section 3.B by introducing the consumer's preference relation and some of its basic properties. We assume throughout that this preference relation is *rational*, offering a complete and transitive ranking of the consumer's possible consumption choices. We also discuss two properties, *monotonicity* (or its weaker version, *local nonsatiation*) and *convexity*, that are used extensively in the analysis that follows.

Section 3.C considers a technical issue: the existence and continuity properties of utility functions that represent the consumer's preferences. We show that not all preference relations are representable by a utility function, and we then formulate an assumption on preferences, known as *continuity*, that is sufficient to guarantee the existence of a (continuous) utility function.

In Section 3.D, we begin our study of the consumer's decision problem by assuming that there are L commodities whose prices she takes as fixed and independent of her actions (the *price-taking assumption*). The consumer's problem is framed as one of *utility maximization* subject to the constraints embodied in the Walrasian budget set. We focus our study on two objects of central interest: the consumer's optimal choice, embodied in the *Walrasian* (or *market* or *ordinary*) —it demand correspondence, and the consumer's optimal utility value, captured by the *indirect utility function*.

Section 3.E introduces the consumer's *expenditure minimization problem*, which bears a close relation to the consumer's goal of utility maximization. In parallel to our study of the demand correspondence and value function of the utility maximization problem, we study the equivalent objects for expenditure minimization. They are known, respectively, as the *Hicksian* (or *compensated*) *demand correspondence* and the *expenditure function*. We also provide an initial formal examination of the relationship between the expenditure minimization and utility maximization problems.

In Section **??**, we pause for an introduction to the mathematical underpinnings of duality theory. This material offers important insights into the structure of preference-based demand theory. Section **??** may be skipped without loss of continuity in a first reading of the chapter. Nevertheless, we recommend the study of its material.

Section **??** continues our analysis of the utility maximization and expenditure minimization problems by establishing some of the most important results of demand theory. These results develop the fundamental connections between the demand and value functions of the two problems.

In Section **??**, we complete the study of the implications of the preference-based theory of consumer demand by asking how and when we can recover the consumer's underlying preferences from her demand behavior, an issue traditionally known as the *integrability problem*. In addition to their other uses, the results presented in this section tell us that the properties of consumer demand identified in Sections 3.D to **??** as necessary implications of preference-maximizing behavior are also *sufficient* in the sense that any demand behavior satisfying these properties can be rationalized as preference-maximizing behavior.

The results in Sections 3.D to **??** also allow us to compare the implications of the preference-based approach to consumer demand with the choice-based theory studied in Section 2.F. Although the differences turn out to be slight, the two approaches are not equivalent; the choice-based demand theory founded on the weak axiom of revealed preference imposes fewer restrictions on demand than does the preference-based theory studied in this chapter. The extra condition added by the assumption of rational preferences turns out to be the *symmetry* of the Slutsky matrix. As a result, we conclude that satisfaction of the weak axiom does not ensure the existence of a rationalizing preference relation for consumer demand.

Although our analysis in Sections 3.B to **??** focuses entirely on the positive (i.e., descriptive) implications of the preference-based approach, one of the most important benefits of the latter is that it provides a framework for normative, or *welfare*, analysis. In Section **??**, we take a first look at this subject by studying the effects of a price change on the consumer's welfare. In this. connection, we discuss the use of the traditional concept of Marshallian surplus as a measure of consumer welfare.

We conclude in Section **??** by returning to the choice-based approach to consumer demand. We ask whether there is some strengthening of the weak axiom that leads to a choice-based theory of consumer demand equivalent to the preference-based approach. As an answer, we introduce the *strong axiom of revealed preference* and show that it leads to demand behavior that is consistent with the existence of underlying preferences.

Appendix A discusses some technical issues related to the continuity and differentiability of Walrasian demand.

For further reading, see the thorough treatment of classical demand theory offered by Deaton and Muellbauer (1980).

3.B Preference Relations: Basic Properties

In the classical approach to consumer demand, the analysis of consumer behavior begins by specifying the consumer's preferences over the commodity bundles in the consumption set $X \subset \mathbb{R}^L_+$.

The consumer's preferences are captured by a preference relation $\succsim$ (an "at-least-as-good-as" relation) defined on X that we take to be *rational* in the sense introduced in Section 1.B; that is, $\succsim$ is *complete* and *transitive*. For convenience, we repeat the formal statement of this assumption from Definition 1.B.1.[1]

Definition 3.B.1. The preference relation $\succsim$ on X is *rational* if it possesses the following two properties:

(i) *Completeness*: For all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity*: For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

In the discussion that follows, we also use two other types of assumptions about preferences: *desirability* assumptions and *convexity* assumptions.

(i) *Desirability assumptions*. It is often reasonable to assume that larger amounts of commodities are preferred to smaller ones. This feature of preferences is captured in the assumption of monotonicity. For Definition 3.B.2, we assume that the consumption of larger amounts of goods is always feasible in principle; that is, if $x \in X$ and $y \geq x$, then $y \in X$.

Definition 3.B.2. The preference relation $\succsim$ on X is *monotone* if $x \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ and $y \neq x$ imply that $y \succ x$.

The assumption that preferences are monotone is satisfied as long as commodities are "goods" rather than "bads". Even if some commodity is a bad, however, we may still be able to view preferences as monotone because it is often possible to redefine a consumption activity in a way that satisfies the assumption. For example, if one commodity is garbage, we can instead define the individual's consumption over the "absence of garbage".[2]

Note that if $\succsim$ is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities. In contrast, strong monotonicity says that if y is larger than x for some commodity and is no less for any other, then y is strictly preferred to x.

For much of the theory, however, a weaker desirability assumption than monotonicity, known as *local nonsatiation*, actually suffices.

Definition 3.B.3. The preference relation $\succsim$ on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.[3]

The test for locally nonsatiated preferences is depicted in Figure 3.B.1 for the case in which $X = \mathbb{R}^L_+$. It says that for any consumption bundle $x \in \mathbb{R}^L_+$ and any arbitrarily small distance away from x, denoted by $\varepsilon > 0$, there is another bundle $y \in \mathbb{R}^L_+$ within this distance from x that is preferred to x. Note that the bundle y may even have less of every commodity than x, as shown

[1] See Section 1.B for a thorough discussion of these properties.

[2] It is also sometimes convenient to view preferences as defined over the level of goods available for consumption (the stocks of goods on hand), rather than over the consumption levels themselves. In this case, if the consumer can freely dispose of any unwanted commodities, her preferences over the level of commodities on hand are monotone as long as some good is always desirable.

[3] $\|x - y\|$ is the Euclidean distance between points x and y; that is, $\|x - y\| = \left[\sum_{l=1}^{L} (x_l - y_l)^2\right]^{1/2}$

in the figure. Nonetheless, when $X = \mathbb{R}^L_+$ local nonsatiation rules out the extreme situation in which all commodities are bads, since in that case no consumption at all (the point $x = 0$) would be a satiation point.

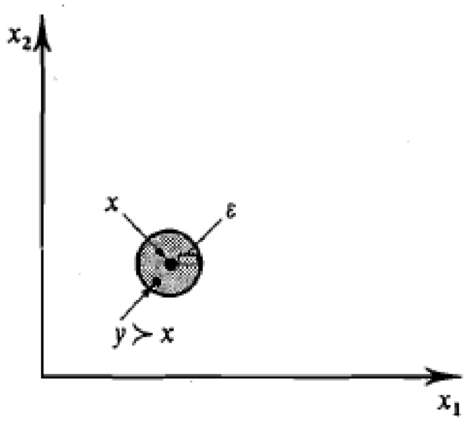

Figure 3.B.1 The test for local nonsatiation.

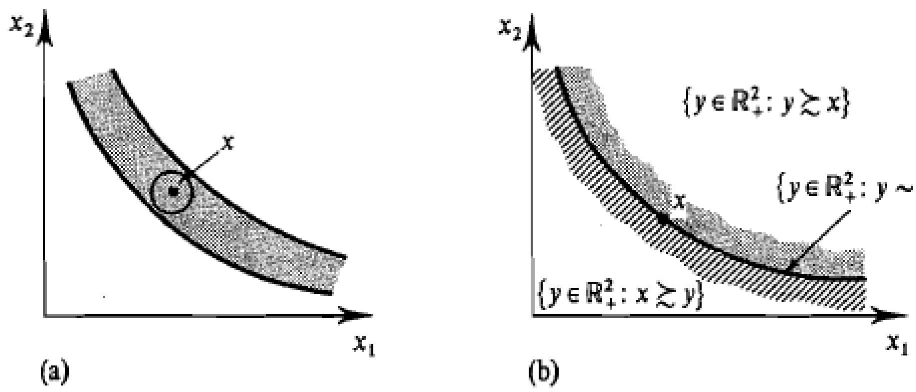

Figure 3.B.2 (a) A thick indifference set violates local nonsatiation. (b) Preferences compatible with local nonsatiation.

Exercise 3.B.1 Show the following:

(a) If $\succsim$ is strongly monotone, then it is monotone.

(b) If $\succsim$ is monotone, then it is locally nonsatiated.

Given the preference relation $\succsim$ and a consumption bundle x, we can define three related sets of consumption bundles. The *indifference set* containing point x is the set of all bundles that are indifferent to x; formally, it is $\{y \in X : y \sim x\}$. The *upper contour set* of bundle x is the set of all bundles that are at least as good as x : $\{y \in X : y \succsim x\}$. The *lower contour set* of x is the set of all bundles that x is at least as good as: $\{y \in X : x \succsim y\}$.

One implication of local nonsatiation (and, hence, of monotonicity) is that it rules out "thick" indifference sets. The indifference set in Figure 3.B.2(a) cannot satisfy local nonsatiation because, if it did, there would be a better point than x within the circle drawn. In contrast, the indifference set in Figure 3.B.2(b) is compatible with local nonsatiation. Figure 3.B.2(b) also depicts the upper and lower contour sets of x.

(ii) *Convexity assumptions*. A second significant assumption, that of convexity of $\succsim$, concerns the trade-offs that the consumer is willing to make among different goods.

Definition 3.B.4. The preference relation $\succsim$ on X is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0,1,]$.

Figure 3.B.3(a) depicts a convex upper contour set; Figure 3.B.3(b) shows an upper contour set that is not convex.

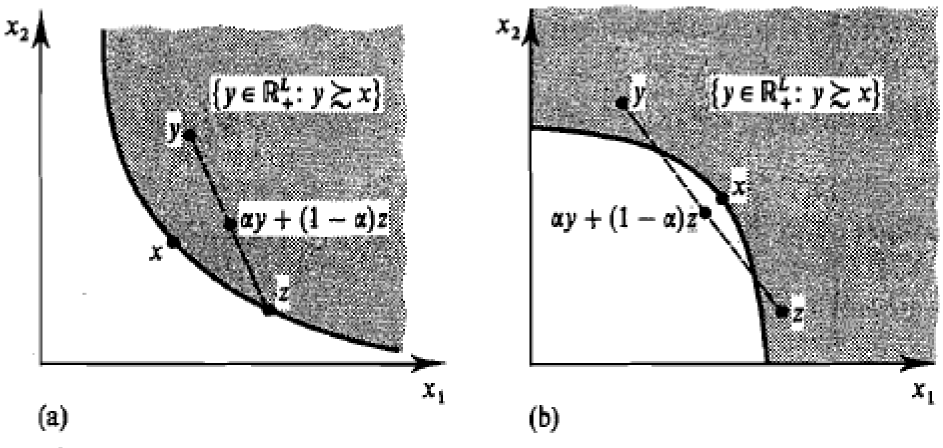

Figure 3.B.3 (a) Convex preferences. (b) Nonconvex preferences.

Convexity is a strong but central hypothesis in economics. It can be interpreted in terms of *diminishing marginal rates of substitution*: That is, with convex preferences, from any initial consumption situation x, and for any two commodities, it takes increasingly larger amounts of one commodity to compensate for successive unit losses of the other.[4]

Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification. Indeed, under convexity, if x is indifferent to y, then $\frac{1}{2}x + \frac{1}{2}y$, the half-half mixture of x and y, cannot be worse than either x or y. In Chapter 6, we shall give a diversification interpretation in terms of behavior under uncertainty. A taste for diversification is a realistic trait of economic life. Economic theory would be in serious difficulty if this postulated propensity for diversification did not have significant descriptive content. But there is no doubt that one can easily think of choice situations where it is violated. For example, you may like both milk and orange juice but get less pleasure from a mixture of the two.

Definition 3.B.4 has been stated for a general consumption set X. But de facto, the convexity assumption can hold only if X is convex. Thus, the hypothesis rules out commodities being consumable only in integer amounts or situations such as that presented in Figure 2.C.3.

[4] More generally, convexity is equivalent to a diminishing marginal rate of substitution between any two goods, provided that we allow for "composite commodities" formed from linear combinations of the L basic commodities.

Although the convexity assumption on preferences may seem strong, this appearance should be qualified in two respects: First, a good number (although not all) of the results of this chapter extend without modification to the nonconvex case. Second, as we show in Appendix A of Chapter 4 and in Section **??**, nonconvexities can often be incorporated into the theory by exploiting regularizing aggregation effects across consumers.

We also make use at times of a strengthening of the convexity assumption.

Definition 3.B.5. The preference relation $\succsim$ on X is *strictly convex* if for every x, we have that $y \succsim x$, $z \succsim x$, and $y \neq z$ implies $\alpha y + (1-\alpha)z \succ x$ for all $\alpha \in (0,1)$.

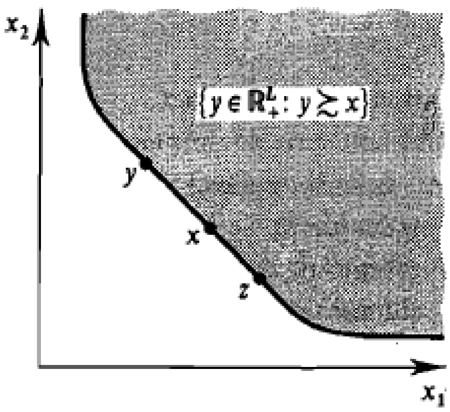

Figure 3.B.4 A convex, but not strictly convex, preference relation.

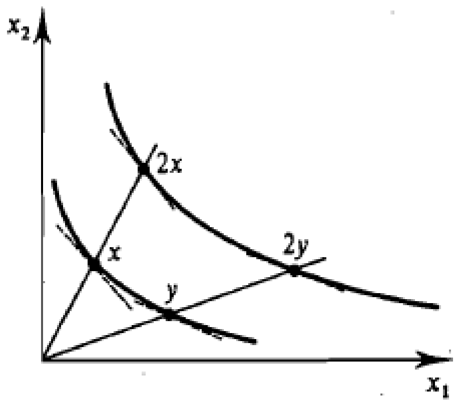

Figure 3.B.5 Homothetic preferences.

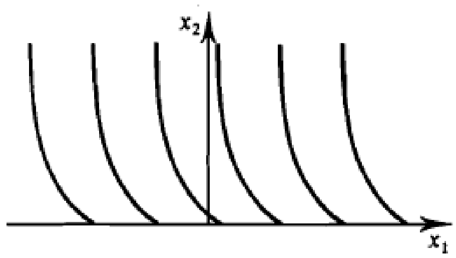

Figure 3.B.6 Quasilinear preferences.

Figure 3.B.3(a) showed strictly convex preferences. In Figure 3.B.4, on the other hand, the preferences, although convex, are not strictly convex.

In applications (particularly those of an econometric nature), it is common to focus on preferences for which it is possible to deduce the consumer's entire preference relation from a single indifference set. Two examples are the classes of *homothetic* and *quasilinear* preferences.

Definition 3.B.6. A monotone preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is *homothetic* if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $a \geq 0$.

Figure 3.B.5 depicts a homothetic preference relation.

Definition 3.B.7. The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is *quasilinear* with respect to commodity 1 (called, in this case, the *numeraire* commodity) if[5]

(i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \cdots, 0)$ and any $\alpha \in \mathbb{R}$.

(ii) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Note that, in Definition 3.B.7, we assume that there is no lower bound on the possible consumption of the first commodity [the consumption set is $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$]. This assumption is convenient in the case of quasilinear preferences (Exercise 3.E will illustrate why). Figure 3.B.6 shows a quasilinear preference relation.

3.C Preference and Utility

For analytical purposes, it is very helpful if we can summarize the consumer's preferences by means of a utility function because mathematical programming techniques can then be used to solve the consumer's problem. In this section, we study when this can be done. Unfortunately, with the assumptions made so far, a rational preference relation need not be representable by a utility function. We begin with an example illustrating this fact and then introduce a weak, economically natural assumption (called *continuity*) that guarantees the existence of a utility representation.

Example 3.C.1. *The Lexicographic Preference Relation.* For simplicity, assume that $X = \mathbb{R}_+^2$. Define $x \succsim y$ if either "$x_1 \succsim y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$" This is known as the *lexicographic preference relation*. The name derives from the way a dictionary is organized; that is, commodity 1 has the highest priority in determining the preference ordering, just as the first letter of a word does in the ordering of a dictionary. When the level of the first commodity in two commodity bundles is the same, the amount of the second commodity in the two bundles determines the consumer's preferences. In Exercise **??**, you are asked to verify that the lexicographic .ordering is complete, transitive, strongly monotone, and strictly convex. Nevertheless, it can be shown that no utility function exists that represents this preference ordering. This is intuitive. With this preference ordering, no two distinct bundles are indifferent; indifference sets are singletons. Therefore, we have two dimensions of distinct indifference sets. Yet, each of these indifference sets must be assigned, in an order-preserving way, a different utility number from the one-dimensional real line. In fact, a somewhat subtle argument is actually required to establish this claim rigorously. It is given, for the more advanced reader, in the following paragraph. ■

[5] More generally, preferences can be quasilinear with respect to any commodity l.

Suppose there is a utility function $u(\cdot)$. For every x_1, we can pick a rational number $r(x_1)$ such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$. Note that because of the lexicographic character of preferences, $x_1 > x_1'$, implies $r(x_1) > r(x_1')$ [since $r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1')$]. Therefore, $r(\cdot)$ provides a one-to-one function from the set of real numbers (which is uncountable) to the set of rational numbers (which is countable). This is a mathematical impossibility. Therefore, we conclude that there can be no utility function representing these preferences.

The assumption that is needed to ensure the existence of a utility function is that the preference relation be continuous.

Definition 3.C.1. The preference relation $\succsim$ on X is *continuous* if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Continuity says that the consumer's preferences cannot exhibit "jumps", with, for example, the consumer preferring each element in sequence x^n to the corresponding element in sequence y^n but suddenly reversing her preference at the limiting points of these sequences x and y.

An equivalent way to state this notion of continuity is to say that for all x, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both *closed*; that is, they include their boundaries. Definition 3.C.1 implies that for any sequence of points $y^{n\infty}_{n=1}$ with $x \succsim y^n$ for all n and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$ (just let $x^n = x$ for all n). Hence, continuity as defined in Definition 3.C.1 implies that the lower contour set is closed; the same is implied for the upper contour set. The reverse argument, that closedness of the lower and upper contour sets implies that Definition 3.C.1 holds, is more advanced and is left as an exercise (Exercise **??**).

Example 3.C.1 (continued)**.** Lexicographic preferences are not continuous. To see this, consider the sequence of bundles $x^n = (l/n, 0)$ and $y^n = (0, l)$. For every n, we have $x^n \succ y^n$. But $\lim_{n\to\infty} y^n = (0, 1) \succ (0, 0) = \lim_{n\to\infty} x^n$. In words, as long as the first component of x is larger than that of y, x is preferred to y even if y_2 is much larger than x_2. But as soon as the first components become equal, only the second components are relevant, and so the preference ranking is reversed at the limit points of the sequence. ■

It turns out that the continuity of $\succsim$ is sufficient for the existence of a utility function representation. In fact, it guarantees the existence of a *continuous* utility function.

Proposition 3.C.1. Suppose that the rational preference relation $\succsim$ on X is *continuous*. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Proof. For the case of $X = \mathbb{R}^L_+$ and a monotone preference relation, there is a relatively simple and intuitive proof that we present here with the help of Figure 3.C.1.

Denote the diagonal ray in $\mathbb{R}^L_+$ (the locus of vectors with all L components equal) by Z. It will be convenient to let e designate the L-vector whose elements are all equal to 1. Then $\alpha e \in Z$ for all nonnegative scalars $\alpha \geq 0$.

Note that for every $x \in \mathbb{R}^L_+$, monotonicity implies that $x \succsim 0$. Also note that for any $\bar{\alpha}$ such that $\bar{\alpha} \gg x$ (as drawn in the figure), we have $\bar{\alpha}e \succsim x$. Monotonicity and continuity can then be shown to imply that there is a unique value $\alpha(x) \in [0, \bar{\alpha}]$ such that $\alpha(x)e \sim x$.

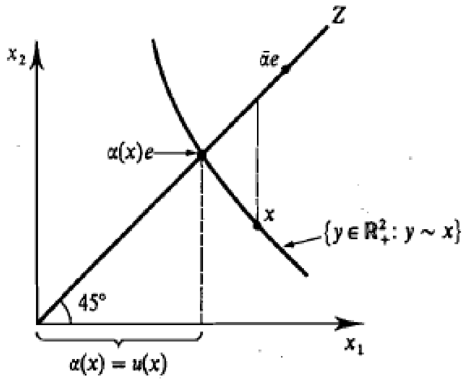

Figure 3.C.1 Construction of a utility function.

Formally, this can be shown as follows: By continuity, the upper and lower contour sets of x are closed. Hence, the sets $A^+ = \alpha \in \mathbb{R}_+ : \alpha e \succsim x$ and $A^- = \alpha \in \mathbb{R}_+ : x \succsim \alpha e$ are nonempty and closed. Note that by completeness of $\succsim$, $\mathbb{R}_+ \subset (A^+ \cup A^-)$. The nonemptiness and closedness of A^+ and A^-, along with the fact that $\mathbb{R}_+$ is connected, imply that $A^+ \cap A^- \neq \varnothing$. Thus, there exists a scalar α such that $\alpha e \succ x$. Furthermore, by monotonicity, $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Hence, there can be at most one scalar satisfying $\alpha e \succ x$. This scalar is $\alpha(x)$.

We now take $\alpha(x)$ as our utility function; that is, we assign a utility value $u(x) = \alpha(x)$ to every x. This utility level is also depicted in Figure 3.C.1. We need to check two properties of this function: that it represents the preference $\succsim$ [i.e., that $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$] and that it is a continuous function. The latter argument is more advanced, and therefore we present it in small type.

That $\alpha(x)$ represents preferences follows from its construction. Formally, suppose first that $\alpha(x) \geq \alpha(y)$. By monotonicity, this implies that $\alpha(x)e \succsim \alpha(y)e$. Since $x \succ \alpha(x)e$ and $y \succ \alpha(y)e$, we have $x \succsim y$. Suppose, on the other hand, that $x \succsim y$. Then $\alpha(x)e \succ x \succsim y \succ \alpha(y)e$; and so by monotonicity, we must have $\alpha(x) \geq \alpha(y)$. Hence, $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$.

We now argue that $\alpha(x)$ is a continuous function at all x; that is, for any sequence $x^{n\infty}_{n=1}$ with $x = \lim_{n\to\infty} x^n$, we have $\lim_{n\to\infty} \alpha(x^n) = \alpha(x)$. Hence, consider a sequence $x^{n\infty}_{n=1}$ such that $x = \lim_{n\to\infty} x^n$.

We note first that the sequence $\{\alpha(x^n)\}_{n=1}^{\infty}$ must have a convergent subsequence. By monotonicity, for any $\varepsilon > 0$, $\alpha(x')$ lies in a compact subset of $\mathbb{R}_+$, $[\alpha_0, \alpha_1]$, for all x^{prime} such that $\|x' - x\| \leq \varepsilon$ (see Figure 3.C.2). Since $x^{n\infty}_{n=1}$ converges to x, there exists an N such that $\alpha(x^n)$ lies in this compact set for all $n > N$. But any infinite sequence that lies in a compact set must have a convergent subsequence (see Section **??** of the Mathematical Appendix).

What remains is to establish that all convergent subsequences of $\{\alpha(x^n)\}_{n=1}^{\infty}$ converge to $\alpha(x)$. To see this, suppose otherwise: that there is some strictly increasing function $m(\cdot)$ that assigns to each

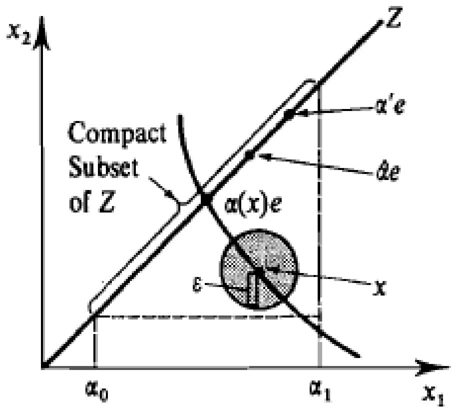

Figure 3.C.2 Proof that the constructed utility function is continuous.

positive integer n a positive integer $m(n)$ and for which the subsequence $\left\{\alpha(x^{m(n)})\right\}_{n=1}^{\infty}$ converges to $\alpha' \neq \alpha(x)$. We first show that $\alpha' > \alpha(x)$ leads to a contradiction. To begin, note that monotonicity would then imply that $\alpha' e \succ \alpha(x)e$. Now, let $\hat{\alpha} = \frac{1}{2}[\alpha' + \alpha(x)]$. The point $\hat{\alpha} e$ is the midpoint on Z between $\hat{\alpha} e$ and $\alpha(x)e$ (see Figure 3.C.2). By monotonicity, $\hat{\alpha} e \succ \alpha(x)e$. Now, since $\alpha\left(x^{m(n)}\right) \to \alpha' > \hat{\alpha}$, there exists an $\bar{N}$ such that for all $n > \bar{N}$, $\alpha\left(x^{m(n)}\right) > \hat{\alpha}$. Hence, for all such n, $x^{m(n)} \succ \alpha\left(x^{m(n)}\right) e \succ \hat{\alpha} e$ (where the latter relation follows from monotonicity). Because preferences are continuous, this would imply that $x \succsim \hat{\alpha} e$. But since $x \succ \alpha(x)e$, we get $\alpha(x)e \succsim \hat{\alpha} e$, which is a contradiction. The argument ruling out $\alpha' < \alpha(x)$ is similar. Thus, since all convergent subsequences of $\{\alpha(x^n)\}_{n=1}^{\infty}$ must converge to $\alpha(x)$, we have $\lim_{n\to\infty} \alpha(x^n) = \alpha(x)$, and we are done. *Q.E.D*

From now on, we assume that the consumer's preference relation is continuous and hence representable by a continuous utility function. As we noted in Section 1.B, the utility function $u(\cdot)$ that represents a preference relation $\succsim$ is not unique; any strictly increasing transformation of $u(\cdot)$, say $v(x) = f(u(x))$, where $f(\cdot)$ is a strictly increasing function, also represents $\succsim$. Proposition 3.C.1 tells us that if $\succsim$ is continuous, there exists *some* continuous utility function representing $\succsim$. But not all utility functions representing $\succsim$ are continuous; any strictly increasing but discontinuous transformation of a continuous utility function also represents $\succsim$.

For analytical purposes, it is also convenient if $u(\cdot)$ can be assumed to be differentiable. It is possible, however, for continuous preferences *not* to be representable by a differentiable utility function. The simplest example, shown in Figure 3.C.3, is the case of *Leontief* preferences, where $x'' \succsim x'$ if and only if $\min x_1'', x_2'' \geq \min x_1', x_2'$. The nondifferentiability arises because of the kink in indifference curves when $x_1 = x_2$.

Whenever convenient in the discussion that follows, we nevertheless assume utility functions to be twice continuously differentiable. It is possible to give a condition purely in terms of preferences that implies this property, but we shall not do so here. Intuitively, what is required is that indifference sets be smooth surfaces that fit together nicely so that the rates at which commodities substitute for each other depend differentiably on the consumption levels.

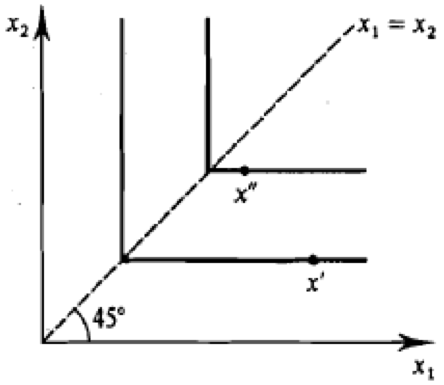

Figure 3.C.3 Leontief preferences cannot be represented by a differentiable utility function.

Restrictions on preferences translate into restrictions on the form of utility functions. The property of monotonicity, for example, implies that the utility function is increasing: $u(x) > u(y)$ if $x \gg y$.

The property of convexity of preferences, on the other hand, implies that $u(\cdot)$ is *quasiconcave* [and, similarly, strict convexity of preferences implies strict quasiconcavity of $u(\cdot)$]. The utility function $u(\cdot)$ is quasiconcave if the set $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all x or, equivalently, if $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$ for any x, y and all $\alpha \in [0,1]$. [If the inequality is strict for all $x \neq y$ and $\alpha \in (0,1)$ then $u(\cdot)$ is strictly quasiconcave; for more on quasiconcavity and strict quasiconcavity see Section **??** of the Mathematical Appendix.] Note, however, that convexity of $\succsim$ does *not* imply the stronger property that $u(\cdot)$ is concave [that $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$ for any x, y and all $\alpha \in [0,1]$]. In fact, although this is a somewhat fine point, there may not be any concave utility function representing a particular convex preference relation $\succsim$.

In Exercise **??**, you are asked to prove two other results relating utility representations and underlying preference relations:

(i) A continuous $\succsim$ on $X = \mathbb{R}^L_+$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one [i.e., such that $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$].

(ii) A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is quasilinear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \cdots, x_L)$.

It is important to realize that although monotonicity and convexity of $\succsim$ imply that all utility functions representing $\succsim$ are increasing and quasiconcave, (i) and (ii) merely say that there is at *least one* utility function that has the specified form. Increasingness and quasiconcavity are *ordinal* properties of $u(\cdot)$; they are preserved for any arbitrary increasing transformation of the utility index. In contrast, the special forms of the utility representations in (i) and (ii) are not preserved; they are —it cardinal properties that are simply convenient choices for a utility representation.[6]

[6]Thus, in this sense, continuity is also a cardinal property of utility functions. See also the discussion of ordinal

3.D The Utility Maximization Problem

We now turn to the study of the consumer's decision problem. We assume throughout that the consumer has a rational, continuous, and locally nonsatiated preference relation, and we take $u(x)$to be a continuous utility function representing these preferences. For the sake of concreteness, we also assume throughout the remainder of the chapter that the consumption set is $X = \mathbb{R}^L_+$.

The consumer's problem of choosing her most preferred consumption bundle given prices $p \gg 0$ and wealth level $w > 0$ can now be stated as the following —it utility maximization problem (UMP):

$$\begin{aligned} \underset{x \geq 0}{Max} \quad & u(x) \\ \text{s.t. } & p \cdot x \leq w \end{aligned}$$

In the UMP, the consumer chooses a consumption bundle in the Walrasian budget set $B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$ to maximize her utility level. We begin with the results stated in Proposition 3.D.1.

Proposition 3.D.1. If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proof. If $p \gg 0$, then the budget set $B_{p,w} = \left\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\right\}$ is a compact set because it is both bounded [for any $l = 1, \cdots, L$, we have $x_l \leq (w/p_l)$ for all $x \in B_{p,w}$] and closed. The result follows from the fact that a continuous function always has a maximum value on any compact set (set Section **??** of the Mathematical Appendix). *Q.E.D*

With this result, we now focus our attention on the properties of two objects that emerge from the UMP: the consumer's set of optimal consumption bundles (the solution set of the UMP) and the consumer's maximal utility value (the value function of the UMP).

The Walrasian Demand Correspondence/Function

The rule that assigns the set of optimal consumption vectors in the UMP to each price-wealth situation $(p, w) \gg 0$ is denoted by $x(p, w) \in \mathbb{R}^L_+$ and is known as the *Walrasian* (or *ordinary* or *market*) *demand correspondence*. An example for $L = 2$ is depicted in Figure 3.D.1(a), where the point $x(p, w)$ lies in the indifference set with the highest utility level of any point in $B_{p,w}$. Note that, as a general matter, for a given $(p, w) \gg 0$ the optimal set $x(p, w)$ may have more than one element, as shown in Figure 3.D.1(b). When $x(p, w)$ is single-valued for all (p, w), we refer to it as the *Walrasian* (or *ordinary* or —it market) *demand function*.[7]

and cardinal properties of utility representations in Section 1.B.

[7]This demand function has also been called the *Marshallian demand function*. However, this terminology can

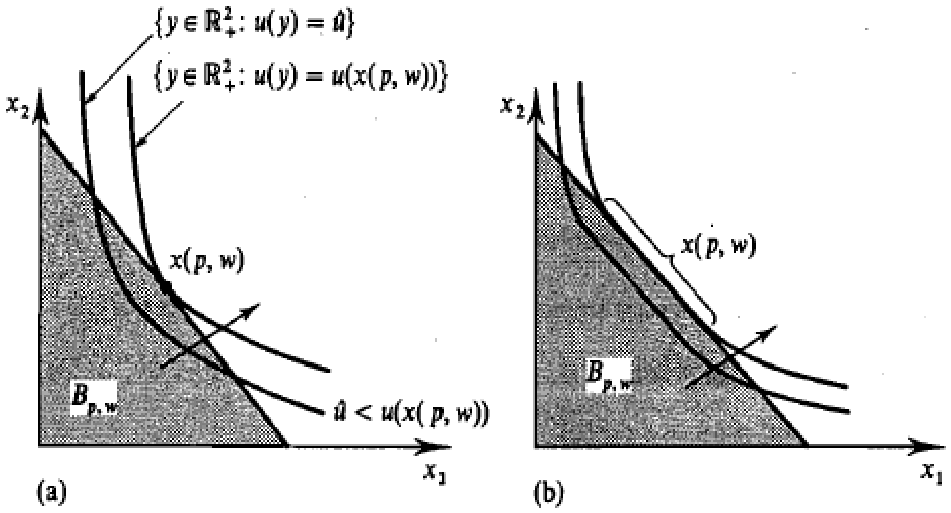

Figure 3.D.1 The utility maximization problem (UMP). (a) Single solution. (b) Multiple solutions.

The properties of $x(p,w)$ stated in Proposition 3.D.2 follow from direct examination of the UMP.

Proposition 3.D.2. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:

(i) *Homogeneity of degree zero in* (p,w) : $x(\alpha p, \alpha w) = x(p,w)$ for any p, w and scalar $\alpha > 0$.

(ii) *Walras' law*: $p \cdot x = w$ for all $x \in x(p,w)$.

(iii) *Convexity/uniqueness*: If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set. Moreover, if $\succsim$ is *strictly convex*, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element.

Proof. We establish each of these properties in turn.

(i) For homogeneity, note that for any scalar a ¿ 0,

$$\{x \in \mathbb{R}^L_+ : \alpha p \cdot x \leq \alpha w\} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$$

that is, the set of feasible consumption bundles in the UMP does not change when all prices and wealth are multiplied by a constant $\alpha > 0$. The set of utility-maximizing consumption bundles must therefore be the same in these two circumstances, and so $x(p,w) = x(\alpha p, \alpha w)$. Note that this property does not require any assumptions on $u(\cdot)$.

(ii) Walras' law follows from local nonsatiation. If $p \cdot x < w$ for some $x \in x(p,w)$, then there must exist another consumption bundle y sufficiently close to x with both $p \cdot y < w$ and $y \succ x$ (see Figure 3.D.2). But this would contradict x being optimal in the UMP.

create confusion, and so we do not use it here. In Marshallian partial equilibrium analysis (where wealth effects are absent), all the different kinds of demand functions studied in this chapter coincide, and so it is not clear which of these demand functions would deserve the Marshall name in the more general setting.

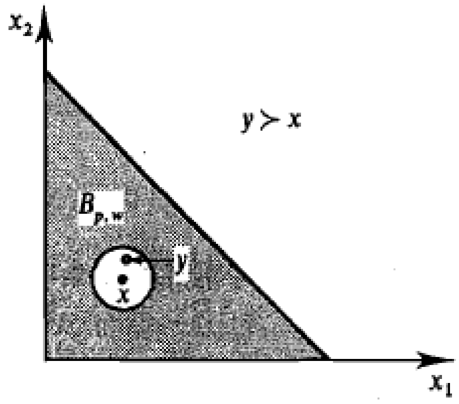

Figure 3.D.2 Local nonsatiation implies Walras' law.

(iii) Suppose that $u(\cdot)$ is quasiconcave and that there are two bundles x and x', with $x \neq x'$, both of which are elements of $x(p,w)$. To establish the result, we show that $x'' = \alpha x + (1-\alpha)x'$ is an element of $x(p,w)$ for any $\alpha \in [O,1]$. To start, we know that $u(x) = u(x')$. Denote this utility level by u^*. By quasiconcavity, $u(x'') \geq u^*$ [see Figure 3.D.3(a)]. In addition, since $p \cdot x \leq w$ and $p \cdot x' \leq w$, we also have

$$p \cdot x'' = p \cdot \left[\alpha x + (1-\alpha)x'\right] \leq w$$

.

Therefore, x'' is a feasible choice in the UMP (put simply, x'' is feasible because $B_{p,w}$ is a convex set). Thus, since $u(x'') \geq u^*$ and x'' is feasible, we have $x'' \in x(p,w)$. This establishes that $x(p,w)$ is a convex set if $u(\cdot)$ is quasiconcave.

Suppose now that $u(\cdot)$ is *strictly* quasiconcave. Following the same argument but using strict quasiconcavity, we can establish that x'' is a feasible choice and that $u(x'') > u^*$ for all $\alpha \in (0,1)$. Because this contradicts the assumption that x and x' are elements of $x(p,w)$, we conclude that there can be at most one element in $x(p,w)$. Figure 3.D.3(b) illustrates this argument. Note the difference from Figure 3.D.3(a) arising from the strict quasiconcavity of $u(x)$. *Q.E.D*

If $u(\cdot)$ is continuously differentiable, an optimal consumption bundle $x^* \in x(p,w)$ can be characterized in a very useful manner by means of first-order conditions. The *Kuhn-Tucker (necessary) conditions* (see Section **??** of the Mathematical Appendix) say that if $x^* \in x(p,w)$ is a solution to the UMP, then there exists a *Lagrange multiplier* $\lambda \geq 0$ such that for all $l = 1,\cdots,L$:[8]

$$\frac{\partial u(x^*)}{\partial c_l} \leq \lambda p_l, \text{ with equality if } x_l^* > 0. \tag{3.D.1}$$

[8]To be fully rigorous, these Kuhn-Tucker necessary conditions are valid only if the constraint qualification condition holds (see Section **??** of the Mathematical Appendix). In the UMP, this is always so. Whenever we use Kuhn-Tucker necessary conditions without mentioning the constraint qualification condition, this requirement is met.

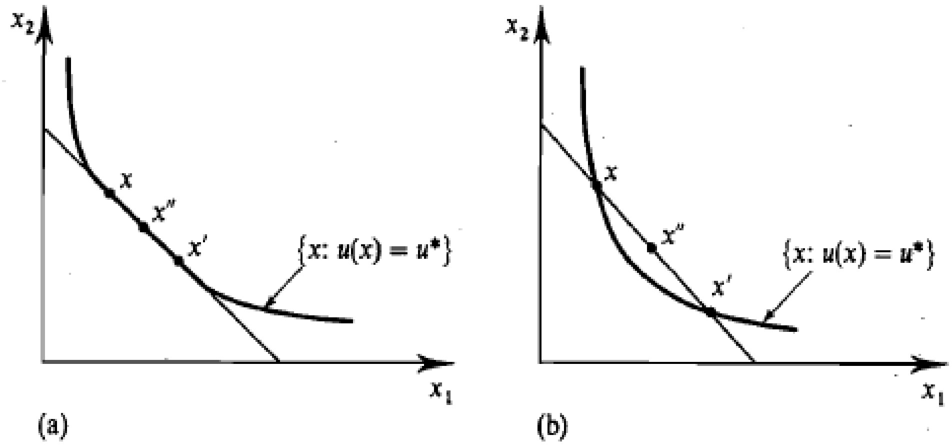

Figure 3.D.3 (a) Convexity of preferences implies convexity of $x(p,w)$. (b) Strict convexity of preferences implies that $x(p,w)$ is single-valued.

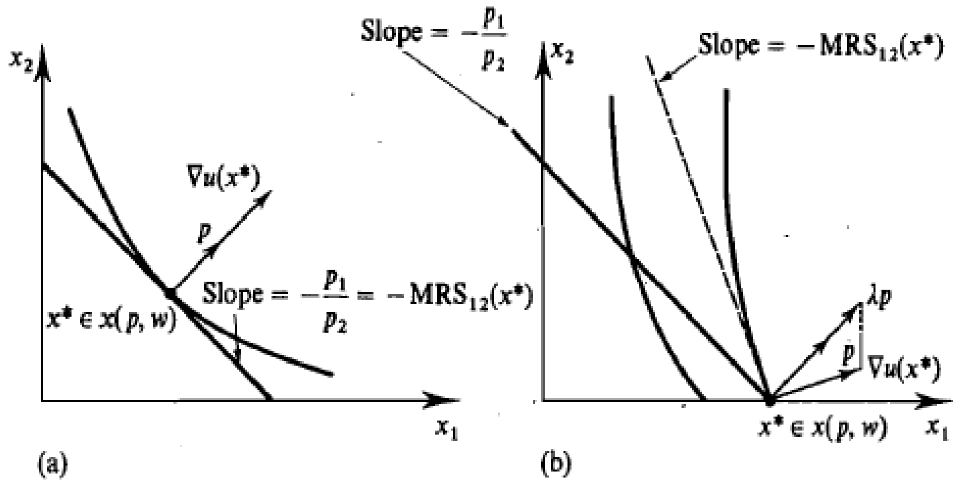

Figure 3.D.4 (a) Interior solution. (b) Boundary solution.

Equivalently, if we let $\nabla u(x) = [\partial u(x)/\partial x_1, \cdots, \partial u(x)/\partial x_L]$ denote the gradient vector of $u(\cdot)$ at x, we can write (3.D.1) in matrix notation as

$$\nabla u(x^*) \leq \lambda p \tag{3.D.2}$$

and

$$x^* \cdot [\nabla u(x^*) - \lambda p] = 0. \tag{3.D.3}$$

Thus, if we are at an interior optimum (i.e., if $x^* \gg 0$), we must have

$$\nabla u(x^*) = \lambda p. \tag{3.D.4}$$

Figure 3.D.4(a) depicts the first-order conditions for the case of an interior optimum when $L = 2$. Condition (3.D.4) tells us that at an interior optimum, the gradient vector of the consumer's utility function $\nabla u(x^*)$ must be proportional to the price vector p, as is shown in Figure

3.D.4(a). If $\nabla u(x^*) \gg 0$, this is equivalent to the requirement that for any two goods l and k, we have

$$\frac{\partial u(x^*)/\partial x_l}{\partial u(x^*)/\partial x_k} = \frac{p_l}{p_k} \tag{3.D.5}$$

The expression on the left of (3.D.5) is the *marginal rate of substitution of good l for good k at x^**, $MRS_{lk}(x^*)$; it tells us the amount of good k that the consumer must be given to compensate her for a one-unit marginal reduction in her consumption of good l.[9] In the case where $L = 2$, the slope of the consumer's indifference set at x^* is precisely — $MRS_{12}(x^*)$. Condition (3.D.5) tells us that at an interior optimum, the consumer's marginal rate of substitution between any two goods must be equal to their price ratio, the marginal rate of exchange between them, as depicted in Figure 3.D.4(a). Were this not the case, the consumer could do better by marginally changing her consumption. For example, if $[\partial u(x^*)/\partial x_l]\,/\,[\partial u(x^*)/\partial x_k] > (p_l/p_k)$, then an increase in the consumption of good l of dx_l, combined with a decrease in good k's consumption equal to $(p_l/p_k)dx_l$, would be feasible and would yield a utility change of $[\partial u(x^*)/\partial x_l]\,dx_l - [\partial u(x^*)/\partial x_k]\,(p_l/p_k)dx_l > 0$.

Figure 3.D.4(b) depicts the first-order conditions for the case of $L = 2$ when the consumer's optimal bundle x^* lies on the boundary of the consumption set (we have $x_2^* = 0$ there). In this case, the gradient vector need not be proportional to the price vector. In particular, the first-order conditions tell us that $\partial u_l(x^*)/\partial x_l \le \lambda p_l$ for those l with $x_l^* = 0$ and $\partial u_l(x^*)/\partial x_l = \lambda p_l$ for those l with $x_L^* > 0$. Thus, in the figure, we see that $MRS_{12}(x^*) > p_1/p_2$. In contrast with the case of an interior optimum, an inequality between the marginal rate of substitution and the price ratio can arise at a boundary optimum because the consumer is unable to reduce her consumption of good 2 (and correspondingly increase her consumption of good 1) any further.

The Lagrange multiplier λ in the first-order conditions (3.D.2) and (3.D.3) gives the marginal, or shadow, value of relaxing the constraint in the UMP (this is a general property of Lagrange multipliers; see Sections **??** and **??** of the Mathematical Appendix). It therefore equals the consumer's marginal utility value of wealth at the optimum. To see this directly, consider for simplicity the case where $x(p,w)$ is a differentiable function and $x(p,w) \gg 0$. By the chain rule, the change in utility from a marginal increase in w is given by $\nabla u(x(p,w)) \cdot D_w x(p,w)$, where $D_w x(p,w) = [\partial x_1(p,w)/\partial w, \cdots, \partial x_L(p,w)/\partial w]$. Substituting for $\nabla u(x(p,w))$ from condition (3.D.4), we get

$$\nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda p \cdot D_w x(p,w) = \lambda,$$

where the last equality follows because $p \cdot x(p,w) = w$ holds for all w (Walras' law) and therefore $p \cdot D_w x(p,w) = 1$. Thus, the marginal change. in utility arising from a marginal increase in wealth—the consumer's *marginal utility of wealth*—is precisely λ.[10]

[9] Note that if utility is unchanged with differential changes in x_l and x_k, dx_l and dx_k, then $[\partial u(x)/\partial x_l]\,dx_l + [\partial u(x)/\partial x_k]\,dx_k = 0$. Thus, when x_l falls by amount $dx_l < 0$. the increase required in x_k to keep utility unchanged is precisely $dx_k = MRS_{lk}(x^*)(-dx_l)$.

[10] Note that if monotonicity of $u(\cdot)$ is strengthened slightly by requiring that $\nabla u(x) \ge 0$ *and* $\nabla u(x) \ne 0$ for all x, then condition (3.D.4) and $p \gg 0$ also imply that λ is strictly positive at any solution of the UMP.

We have seen that conditions (3.D.2) and (3.D.3) must necessarily be satisfied by any $x^* \in (p,w)$. When, on the other hand, does satisfaction of these first-order conditions by some bundle x imply that x is a solution to the UMP? That is, when are the first-order conditions *sufficient* to establish that x is a solution? If $u(\cdot)$ is quasiconcave and monotone and has $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}^L_+$, then the Kuhn-Tucker first-order conditions are indeed sufficient (see Section **??** of the Mathematical Appendix). What if $u(\cdot)$ is not quasiconcave? In that case, if $u(\cdot)$ is locally quasiconcave at x^*, and if x^* satisfies the first-order conditions, then x^* is a local maximum. Local quasiconcavity can be verified by means of a determinant test on the *bordered Hessian matrix* of $u(\cdot)$ at x^*. (For more on this, see Sections **??** and **??** of the Mathematical Appendix.)

Example 3.D.1 illustrates the use of the first-order conditions in deriving the consumer's optimal consumption bundle.

Example 3.D.1. *The Demand Function Derived from the Cobb-Douglas Utility Function.* A *Cobb-Douglas* utility function for $L=2$ is given by $u(x_1,x_2) = kx_1^{\alpha}x_2^{1-\alpha}$ for some $\alpha \in (0,1)$ and $k>0$. It is increasing at all $(x_1,x_2) \gg 0$ and is homogeneous of degree one. For our analysis, it turns out to be easier to use the increasing transformation $\alpha \ln x_1 + (1-\alpha)\ln x_2$, a strictly concave function, as our utility function. With this choice, the UMP can be stated as

$$\begin{aligned} \underset{x_1,x_2}{Max} \quad & \alpha \ln x_1 + (1-\alpha)\ln x_2 \\ & \text{s.t. } p_1x_1 + p_2x_2 = w. \end{aligned} \tag{3.D.6}$$

[Note that since $u(\cdot)$ is increasing, the budget constraint will hold with strict equality at any solution.]

Since $\ln 0 = -\infty$, the optimal choice $(x_1(p,w), x_2(p,w))$ is strictly positive and must satisfy the first-order conditions (we write the consumption levels simply as x_1 and x_2 for notational convenience)

$$\frac{\alpha}{x_1} = \lambda p_1 \tag{3.D.7}$$

and

$$\frac{1-\alpha}{x_2} = \lambda p_2 \tag{3.D.8}$$

for some $\lambda \geq 0$, and the budget constraint $p \cdot x(p,w) = w$. Conditions (3.D.7) and (3.D.8) imply that

$$p_1x_1 = \frac{\alpha}{1-\alpha} p_2x_2$$

or, using the budget constraint,

$$p_1x_1 = \frac{\alpha}{1-\alpha}(w - p_1x_1). \tag{3.D.9}$$

Hence (including the arguments of x_1 and x_2 once again)

$$x_1(p,w) = \frac{\alpha w}{p_1}$$

and (using the budget constraint)

$$x_2(p,w) = \frac{(1-\alpha)w}{p_2}$$

Note that with the Cobb-Douglas utility function, the expenditure on each commodity is a constant fraction of wealth for any price vector p [a share of α goes for the first commodity and a share of $(1-\alpha)$ goes for the second]. ■

Exercise 3.D.1 Verify the three properties of Proposition 3.D.2 for the Walrasian demand function generated by the Cobb-Douglas utility function.

For the analysis of demand responses to changes in prices and wealth, it is also very helpful if the consumer's Walrasian demand is suitably continuous and differentiable. Because the issues are somewhat more technical, we will discuss the conditions under which demand satisfies these properties in Appendix A to this chapter. We conclude there that both properties hold under fairly general conditions. Indeed, if preferences are continuous, strictly convex, and locally nonsatiated on the. consumption set $\mathbb{R}^L_+$, then $x(p,w)$ (which is then a function) is *always* continuous at all $(p,w) \gg 0$.

The Indirect Utility Function

For each $(p,w) \gg 0$, the utility value of the UMP is denoted $v(p,w) \in \mathbb{R}$. It is equal to $u(x^*)$ for any $x^* \in x(p,w)$. The function $v(p,w)$ is called the *indirect utility function* and often proves to be a very useful analytic tool. Proposition 3.D.3 identifies its basic properties.

Proposition 3.D.3. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p,w)$ is

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_l for any l.

(iii) Quasiconvex; that is, the set $\{(p,w) : v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$.[11]

Proof. Except for quasiconvexity and continuity all the properties follow readily from our previous discussion. We forgo the proof of continuity here but note that, when preferences are strictly convex, it follows from the fact that $x(p,w)$ and $u(x)$ are continuous functions because $v(p,w) = u(x(p,w))$ [recall that the continuity of $x(p,w)$ is established in Appendix A of this chapter].

To see that $v(p,w)$ is quasiconvex, suppose that $v(p,w) \leq \bar{v}$ and $v(p',w') \leq \bar{v}$. For any $\alpha \in [0,1]$, consider then the price-wealth pair $(p'',w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$.

[11] Note that property (iii) says that $v(p,w)$ is quasi*convex*, not quasi—it concave. Observe also that property (iii) does not require for its validity that $u(\cdot)$ be quasiconcave.

To establish quasiconvexity, we want to show that $v(p'', w'') \leq \bar{v}$. Thus, we show that for any x with $p'' \cdot x \leq w''$, we must have $u(x) \leq \bar{v}$. Note, first, that if $p'' \cdot x \leq w''$, then,

$$\alpha p \cdot x + (1-\alpha) p' \cdot x \leq \alpha w + (1-\alpha) w'.$$

Hence, either $p \cdot x \leq w$ or $p' \cdot x \leq w'$ (or both). If the former inequality holds, then $u(x) \leq v(p, w) \leq \bar{v}$, and we have established the result. If the latter holds, then $u(x) \leq v(p', w') \leq \bar{v}$, and the same conclusion follows. *Q.E.D*

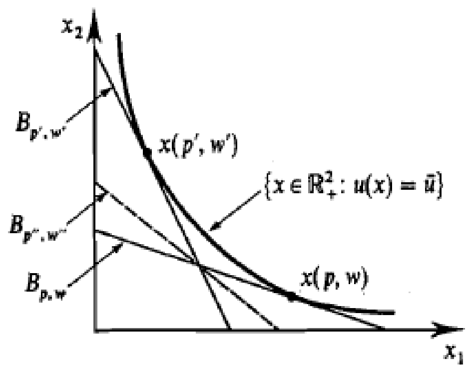

Figure 3.D.5 The indirect utility function $v(p, w)$ is quasiconvex.

The quasiconvexity of $v(p, w)$ can be verified graphically in Figure 3.D.5 for the case where $L = 2$. There, the budget sets for price-wealth pairs (p, w) and (p', w') generate the same maximized utility value $\bar{u}$. The budget line corresponding to $(p'', w'') = (\alpha p + (1-\alpha) p', \alpha w + (1-\alpha) w')$ is depicted as a dashed line in Figure 3.D.5. Because (p'', w'') is a convex combination of (p, w) and (p', w'), its budget line lies between the budget lines for these two price-wealth pairs. As can be seen in the figure, the attainable utility under (p'', w'') is necessarily no greater than $\bar{u}$.

Note that the indirect utility function depends on the utility representation chosen. In particular, if $v(p, w)$ is the indirect utility function when the consumer's utility function is $u(\cdot)$, then the indirect utility function corresponding to utility representation $\tilde{u}(x) = f(u(x))$ is $\tilde{v}(p, w) = f(v(p, w))$.

Example 3.D.2. Suppose that we have the utility function $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha) \ln x_2$. Then, substituting $x_1(p, w)$ and $x_2(p, w)$ from Example 3.D.1, into $u(x)$ we have

$$\begin{aligned} v(p, w) &= u(x(p, w)) \\ &= [\alpha \ln \alpha + (1-\alpha) \ln(1-\alpha)] + \ln w - \alpha \ln p_1 - (1-\alpha) \ln p_2. \end{aligned}$$

■

Exercise 3.D.2 Verify the four properties of Proposition 3.D.3 for the indirect utility function derived in Example 3.D.2.

3.E The Expenditure Minimization Problem

In this section, we study the following *expenditure minimization problem* (EMP) for $p \gg 0$ and $u > u(0)$:[12]

$$\begin{aligned} \underset{x \geq 0}{Min} \quad & p \cdot x \\ \text{s.t. } & u(x) \geq u. \end{aligned}$$

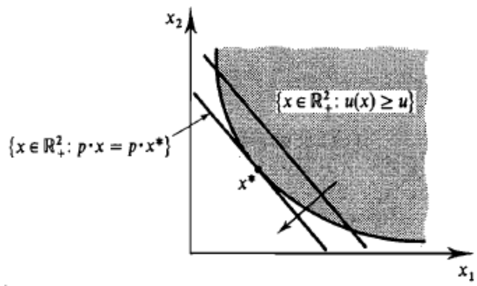

Figure 3.E.1 The expenditure minimization problem (EMP).

Whereas the UMP computes the maximal level of utility that can be obtained given wealth w, the EMP computes the minimal level of wealth required to reach utility level u. The EMP is the "dual" problem to the UMP. It captures the same aim of efficient use of the consumer's purchasing power while reversing the roles of objective function and constraint.[13]

Throughout this section, we assume that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $\mathbb{R}^L_+$.

The EMP is illustrated in Figure 3.E.1. The optimal consumption bundle x^* is the least costly bundle that still allows the consumer to achieve the utility level u. Geometrically, it is the point in the set $\{x \in \mathbb{R}^L_+ : u(x) \geq u\}$ that lies on the lowest possible budget line associated with the price vector p.

Proposition 3.E.1 describes the formal relationship between EMP and the UMP.

Proposition 3.E.1. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

(i) if x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.

[12] Utility $u(0)$ is the utility from consuming the consumption bundle $x = (0, 0, \cdots, 0)$. The restriction to $u > u(0)$ rules out only uninteresting situations.

[13] The term "dual" is meant to be suggestive. It is usually applied to pairs of problems and concepts that are formally similar except that the role of quantities and prices, and/or maximization and minimization, and/or objective function and constraint, have been reversed.

(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the LIMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proof. (i)Suppose that x^* is not optimal in the EMP with required utility level $u(x^*)$. Then there exists an x' such that $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^* \leq w$. By local nonsatiation, we can find an x'' very close to x' such that $u(x'') > u(x')$ and $p \cdot x'' < w$. But this implies that $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$, contradicting the optimality of x^* in the UMP. Thus, x^* must be optimal in the EMP when the required utility level is $u(x^*)$, and the minimized expenditure level is therefore $p \cdot x^*$. Finally, since * solves the UMP when wealth is w, by Walras' law we have $p \cdot x^* = w$.

(ii)Since $u > u(0)$, we must have $x^* \neq 0$. Hence, $p \cdot x^* > 0$. Suppose that x^* is not optimal in the UMP when wealth is $p \cdot x^*$. Then there exists an x' such that $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Consider a bundle $x'' = \alpha x'$ where $\alpha \in (0,1)$ (x'' is a "scaled-down" version of x'). By continuity of $u(\cdot)$, if α is close enough to 1, then we will have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. But this contradicts the optimality of x^* in the EMP. Thus, x^* must be optimal in the UMP when wealth is $p \cdot x^*$, and the maximized utility level is therefore $u(x^*)$. In Proposition **??**(ii), we will show that if x^* solves the EMP when the required utility level is u, then $u(x^*) = u$. *Q.E.D*

As with the UMP, when $p \gg 0$ a solution to the EMP exists under very general conditions. The constraint set merely needs to be nonempty; that is, $u(\cdot)$ must attain values at least as large as u for *some* x (see Exercise 3.E). From now on, we assume that this is so; for example, this condition will be satisfied for any $u > u(0)$ if $u(\cdot)$ is unbounded above.

We now proceed to study the optimal consumption vector and the value function of the EMP. We consider the value function first.

EXERCISES

Exercise 3.D.1 A consumer lives for two periods, denoted 1 and 2, and consumes a single consumption good in each period. His wealth when born is $w > 0$. What is his (lifetime) Walrasian budget set?

Exercise 3.D.2 A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumer's Walrasian budget set?

Exercise 3.D.3 Consider an extension of the Walrasian budget set to an arbitrary consumption set $X : B_{p,w} = \{x \in X : p \cdot x \leq w\}$. Assume $(p,w) \gg 0$.

(a) If X is the set depicted in Figure 2.C.3, would $B_{p,w}$ be convex?

(b) Show that if X is a convex set, then $B_{p,w}$ is as well.

Exercise 3.D.4 Show that the budget set in Figure 3.D.4 is not convex.

Exercise 3.E.1 In text.

Exercise 3.E.2 In text.

Exercise 3.E.3 Use Propositions 2.E.1 to 2.E.3 to show that $p \cdot D_p x(p,w) p = -w$. Interpret.

Exercise 3.E.4 Show that if $x(p,w)$ is homogeneous of degree one with respect to w [ie., $x(p,\alpha w) = \alpha x(p,w)$ for all $\alpha > 0$] and satisfies Walras' law, then $\varepsilon_{lw}(p,w) = 1$ for every l. Interpret. Can you say something about $D_w x(p,w)$ and the form of the Engel functions and curves in this case?

Exercise 3.E.5 Suppose that $x(p,w)$ is a demand function which is homogeneous of degree one with respect to w and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is $\partial x_l(p,w)/\partial p_k = 0$ whenever $k \neq l$. Show that this implies that for every $l, x_l(p,w) = \alpha w/p_l$, where $\alpha_l > 0$ is a constant independent of (p,w).

Exercise 3.E.6 Verify that the conclusions of Propositions 2.E.1 to 2.E.3 hold for the demand function given in Exercise 2.E when $\beta = 1$.

Exercise 3.E.7 A consumer in a two-good economy has a demand function $x(p,w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p,w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

Exercise 3.E.8 Show that the elasticity of demand for good l with respect to price $p_k, \varepsilon_{lk}(p,w)$, can be written as $\varepsilon_{lk}(p,w) = d\ln(x_l(p,w))/d\ln(p_k)$, where $\ln(\cdot)$ is the natural logarithm function. Derive a similar expression for $\varepsilon_{lw}(p,w)$. Conclude that if we estimate the parameters $(\alpha_0, \alpha_1, \alpha_2, \gamma)$ of the equation $\ln(x_l(p,w)) = \alpha_0, \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$, these parameter estimates provide us with estimates of the elasticities $\varepsilon_{l1}(p,w)$, $\varepsilon_{l2}(p,w)$, and $\varepsilon_{lw}(p,w)$.

Exercise 3.F.1 Show that for Walrasian demand functions, the definition of the weak axiom given in Definition 2.F.1 coincides with that in Definition 1.C.1.

Exercise 3.F.2 Verify the claim of Example 2.F.1.

Exercise 3.F.3 You are given the following partial information about a consumer's purchases. He consumes only two goods.

	Year1		Year2	
	Quantity	Price	Quantity	Price
Good 1	100	100	120	100
Good 2	100	100	?	80

Over what range of quantities of good 2 consumed in year 2 would you conclude:

(a) That his behaviour is inconsistent (i.e., in contradiction with the weak axiom)?

(b) That the consumer's consumption bundle in year 1 is revealed preferred to that in year 2?

(c) That the consumer's consumption bundle in year 2 is revealed preferred to that in year 1?

(d) That there is insufficient information to justify (a), (b), and/or (c)?

(e) That good 1 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.

(f) That good 2 is an inferior good (at some price) for this consumer? Assume that the weak axiom is satisfied.

Exercise 3.F.4 Consider the consumption of a consumer in two different periods, period 0 and period 1. Period t prices, wealth, and consumption are p^t, w_t, and $x^t = x(p^t, w_t,)$, respectively. It is often of applied interest to form an index measure of the quantity consumed by a consumer. The *Laspeyres* quantity index computes the change in quantity using period 0 prices as weights: $L_Q = (p^0 \cdot x^1)/(p^0 \cdot x^0)$. The *Paasche* quantity index instead uses period 1 prices as weights: $P_Q = (p^1 \cdot x^1)/(p^1 \cdot x^0)$. Finally, we could use the consumer's expenditure change: $E_Q = (p^1 \cdot x^1)/(p^0 \cdot x^0)$. Show the following:

(a) If $L_Q < 1$, then the consumer has a revealed preference for x^0 over x^1.

(b) If $P_Q > 1$, then the consumer has a revealed preference for x^1 over x^0.

(c) No revealed preference relationship is implied by either $E_Q > 1$ or $E_Q < 1$. Note that at the aggregate level, E_Q corresponds to the percentage change in gross national product.

Exercise 3.F.5 Suppose that $x(p,w)$ is a differentiable demand function that satisfies the weak axiom, Walras' law, and homogeneity of degree zero. Show that if $x(\cdot,\cdot)$ is homogeneous of degree one with respect to w [i.e., $x(p,\alpha w) = \alpha x(p,w)$ for all (p,w) and $\alpha > 0$], then the law of demand holds even for uncompensated price changes. If this is easier, establish only the infinitesimal version of this conclusion; that is, $dp \cdot D_p x(p,w) dp \leq 0$ for any dp.

Exercise 3.F.6 Suppose that $x(p,w)$ is homogeneous of degree zero. Show that the weak axiom holds if and only if for some $w > 0$ and all p,p' we have $p' \cdot x(p,w) > w$ whenever $p \cdot x(p',w) \leq w$ and $x(p',w) \neq x(p,w)$.

Exercise 3.F.7 In text.

Exercise 3.F.8 Let $\hat{s}_{lk}(p,w) = [p_k/x_l(p,w)]\, s_{li}(p,w)$ be the substitution terms in elasticity form. Express $\hat{s}_{lk}(p,w)$ in terms of $\varepsilon_{lk}(p,w)$, $\varepsilon_{lw}(p,w)$, and $b_k(p,w)$.

Exercise 3.F.9 A symmetric $n \times n$ matrix A is negative definite if and only if $(-1)^k |A_{kk}| > 0$ for all $k \leq n$, where A_{kk}, is the submatrix of A obtained by deleting the last $n-k$ rows and columns. For semidefiniteness of the symmetric matrix A, we replace the strict inequalities by weak inequalities and require that the weak inequalities hold for all matrices formed by permuting the rows and columns of A (see Section M.D of the Mathematical Appendix for details).

(a) Show that an arbitrary (possibly nonsymmetric) matrix A is negative definite (or semidefinite) if and only if $A + A^T$ is negative definite (or semidefinite). Show also that the above determinant condition (which can be shown to be necessary) is no longer sufficient in the nonsymmetric case.

(b) Show that for $L = 2$, the necessary and sufficient condition for the substitution matrix $S(p,w)$ of rank 1 to be negative semidefinite is that any diagonal entry (i.e., any own-price substitution effect) be negative.

Exercise 3.F.10 Consider the demand function in Exercise 2.E with $\beta = 1$. Assume that $w = 1$.

(a) Compute the substitution matrix. Show that at $p = (1,1,1)$, it is negative semidefinite but not symmetric.

(b) Show that this demand function does not satisfy the weak axiom. [*Hint*: Consider the price vector $p = (1, 1, \varepsilon)$ and show that the substitution matrix is not negative semidefinite (for $\varepsilon > 0$ small).]

Exercise 3.F.11 Show that for $L = 2$, $S(p, w)$ is always symmetric. [*Hint*: Use Proposition **??**.]

Exercise 3.F.12 Show that if the Walrasian demand function $x(p, w)$ is generated by a rational preference relation, than it must satisfy the weak axiom.

Exercise 3.F.13 Suppose that $x(p, w)$ may be multivalued.

(a) From the definition of the weak axiom given in Section 1.C, develop the generalization of Definition 2.F.1 for Walrasian demand correspondences.

(b) Show that if $x(p, w)$ satisfies this generalization of the weak axiom and Walras' law, then $x(\cdot)$ satisfies the following property:

$$(*) \text{ For any } x \in x(p, w) \text{ and } x' \in x(p', w'), \text{ if } p \cdot x' < w, \text{ then } p \cdot x > w.$$

(c) Show that the generalized weak axiom and Walras' law implies the following generalized version of the compensated law of demand: Starting from any initial position (p, w) with demand $x \in x(p, w)$, for any compensated price change to new prices p' and wealth level $w' = p' \cdot x$, we have

$$(p' - p) \cdot (x' - x) \leq 0$$

for all $x' \in x(p', w')$, with strict inequality if $x' \in x(p, w)$.

(d) Show that if $x(p, w)$ satisfies Walras' law and the generalized compensated law of demand defined in (c), then $x(p, w)$ satisfies the generalized weak axiom.

Exercise 3.F.14 Show that if $x(p, w)$ is a Walrasian demand function that satisfies the weak axiom, then $x(p, w)$ must be homogeneous of degree zero.

Exercise 3.F.15 Consider a setting with $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. The consumer's demand function $x(p, w)$ satisfies homogeneity of degree zero, Walras' law and (fixing $p_3 = 1$) has

$$x_1(p, w) = -p_1 + p_2$$

and

$$x_2(p, w) = -p_2$$

Show that this demand function satisfies the weak axiom by demonstrating that its substitution matrix satisfies $v \cdot S(p, w)v < 0$ for all $v \neq \alpha p$. [*Hint*: Use the matrix results recorded in Section M.D of the Mathematical Appendix.] Observe then that the substitution matrix is not symmetric. (*Note*: The fact that we allow for negative consumption levels here is not essential for finding a demand function that satisfies the weak axiom but whose substitution matrix is not symmetric; with a consumption set allowing only for nonnegative consumption levels, however, we would need to specify a more complicated demand function.)

Exercise 3.F.16 Consider a setting where $L = 3$ and a consumer whose consumption set is $\mathbb{R}^3$. Suppose that his demand function $x(p,w)$ is

$$x_1(p,w) = \frac{p_2}{p_3},$$
$$x_2(p,w) = -\frac{p_1}{p_3},$$
$$x_3(p,w) = \frac{w}{p_3}.$$

(a) Show that $x(p,w)$ is homogeneous of degree zero in (p,w) and satisfies Walras' law.

(b) Show that $x(p,w)$ violates the weak axiom.

(c) Show that $v \cdot S(p,w)v = 0$ for all $v \in \mathbb{R}^3$.

Exercise 3.F.17 In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p,w) = \frac{w}{\left(\sum\limits_{l=1}^{L} p_l\right)} \text{ for } k = 1, \cdots, L.$$

(a) Is this demand function homogeneous of degree zero in (p,w)?

(b) Does it satisfy Walras' law?

(c) Does it satisfy the weak axiom?

(d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

Bibliography

Chapter 4
Aggregate Demand

4.A Introduction

Bibliography

Chapter 5
Production

5.A Introduction

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Chapter 6
Choice Under Uncertainty

6.A Introduction

In previous chapters, we studied choices that result in perfectly certain outcomes. In reality, however, many important economic decisions involve an element of risk. Although it is formally possible to analyze these situations using the general theory of choice developed in Chapter 1, there is good reason to develop a more specialized theory: Uncertain alternatives have a structure that we can use to restrict the preferences that "rational" individuals may hold. Taking advantage of this structure allows us to derive stronger implications than those based solely on the framework of Chapter 1.

In Section 6.B, we begin our study of choice under uncertainty by considering a setting in which alternatives with uncertain outcomes are describable by means of objectively known probabilities defined on an abstract set of possible outcomes. These representations of risky alternatives are called *lotteries*. In the spirit of Chapter 1, we assume that the decision maker has a rational preference relation over these lotteries. We then proceed to derive the *expected utility theorem*, a result of central importance. This theorem says that under certain conditions, we can represent preferences by an extremely convenient type of utility function, one that possesses what is called the *expected utility form*. The key assumption leading to this result is the *independence axiom*, which we discuss extensively.

In the remaining sections, we focus on the special case in which the outcome of a risky choice is an amount of money (or any other one-dimensional measure of consumption). This case underlies much of finance and portfolio theory, as well as substantial areas of applied economics.

In Section 6.C, we present the concept of *risk aversion* and discuss its measurement. We then study the comparison of risk aversions both across different individuals and across different levels of an individual's wealth.

Section 6.D is concerned with the comparison of alternative distributions of monetary returns. We ask when one distribution of monetary returns can unambiguously be said to be "better" than another, and also when one distribution can be said to be "more risky than" another. These comparisons lead, respectively, to the concepts of *first-order* and *second-order stochastic dominance*.

In Section 6.E, we extend the basic theory by allowing utility to depend on *states of nature* underlying the uncertainty as well as on the monetary payoffs. In the process, we develop a framework for modeling uncertainty in terms of these underlying states. This framework is often of great analytical convenience, and we use it extensively later in this book.

In Section 6.F, we consider briefly the theory of *subjective probability*. The assumption that uncertain prospects are offered to us with known objective probabilities, which we use in Section 6.B to derive the expected utility theorem, is rarely descriptive of reality. The subjective probability framework offers a way of modeling choice under uncertainty in which the probabilities of different risky alternatives are not given to the decision maker in any objective fashion. Yet, as we shall see, the theory of subjective probability offers something of a rescue for our earlier objective probability approach.

For further reading on these topics, see Kreps (1988) and Machina (1987). Diamond and Rothschild (1978) is an excellent sourcebook for original articles.

6.B Expected Utility Theory

We begin this section by developing a formal apparatus for modeling risk. We then apply this framework to the study of preferences over risky alternatives and to establish the important expected utility theorem.

Description of Risky Alternatives

Let us imagine that a decision maker faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible *outcomes*, but which outcome will actually occur is uncertain at the time that he must make his choice.

Formally, we denote the set of all possible outcomes by C.[1] These outcomes could take many forms. They could, for example, be consumption bundles. In this case, $C = X$, the decision maker's consumption set. Alternatively, the outcomes might take the simpler form of monetary payoffs. This case will, in fact, be our leading example later in this chapter. Here, however, we treat C as an abstract set and therefore allow for very general outcomes.

To avoid some technicalities, we assume in this section that the number of possible outcomes in C is finite, and we index these outcomes by $n = 1, \cdots, N$.

Throughout this and the next several sections, we assume that the probabilities of the various outcomes arising from any chosen alternative are *objectively known*. For example, the risky alternatives might be monetary gambles on the spin of an unbiased roulette wheel.

The basic building block of the theory is the concept of a *lottery*, a formal device that is used to represent risky alternatives.

Definition 6.B.1. A *simple lottery* L is a list $L = (p_1, \cdots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

A simple lottery can be represented geometrically as a point in the $(N-1)$ *dimensional simplex*, $\Delta = \left\{p \in \mathbb{R}^N_+ : p_1 + \cdots + p_N = 1\right\}$. Figure 6.B.1(a) depicts this simplex for the case in

[1] It is also common, following Savage (1954), to refer to the elements of C as *consequences*.

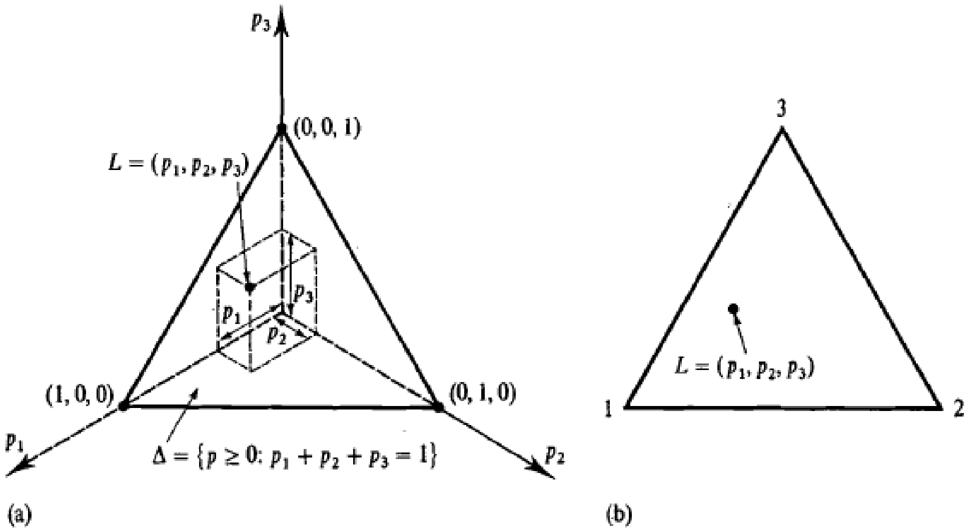

Figure 6.B.1 Representations of the simplex when $N = 3$. (a) Three-dimensional representation. (b) Two-dimensional representation.

which $N = 3$. Each vertex of the simplex stands for the degenerate lottery where one outcome is certain and the other two outcomes have probability zero. Each point in the simplex represents a lottery over the three outcomes. When $N = 3$, it is convenient to depict the simplex in two dimensions, as in Figure 6.B.1(b), where it takes the form of an equilateral triangle.[2]

In a simple lottery, the outcomes that may result are certain. A more general variant of a lottery, known as a *compound lottery*, allows the outcomes of a lottery themselves to be simple lotteries.[3]

Definition 6.B.2. Given K simple lotteries $L_k = (p_1^k, \cdots, p_N^k), k = 1, \cdots, K$, and probabilities $\alpha \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \cdots, K$.

For any compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, we can calculate a corresponding *reduced lottery* as the simple lottery $L = (p_1, \cdots, p_N)$ that generates the same ultimate distribution over outcomes. The value of each p_n is obtained by multiplying the probability that each lottery L_k arises, α_k, by the probability p_n^k that outcome n arises in lottery L_k, and then adding over k. That is, the probability of outcome n in the reduced lottery is

$$p_n = \alpha_1 p_n^1 + \cdots + \alpha_K p_n^K$$

[2] Recall that equilateral triangles have the property that the sum of the perpendiculars from any point to. the three sides is equal to the altitude of the triangle. It is therefore common to depict the simplex when $N = 3$ as an equilateral triangle with altitude equal to 1 because by doing so, we have the convenient geometric property that the probability p_n of outcome n in the lottery associated with some point in this simplex is equal to the length of the perpendicular from this point to the side opposite the vertex labeled n.

[3] We could also define compound lotteries with more than two stages. We do not do so because we will not need them in this chapter. The principles involved, however, are the same.

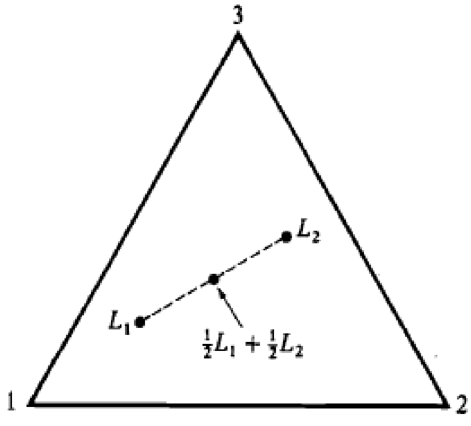

Figure 6.B.2 The reduced lottery of a compound lottery.

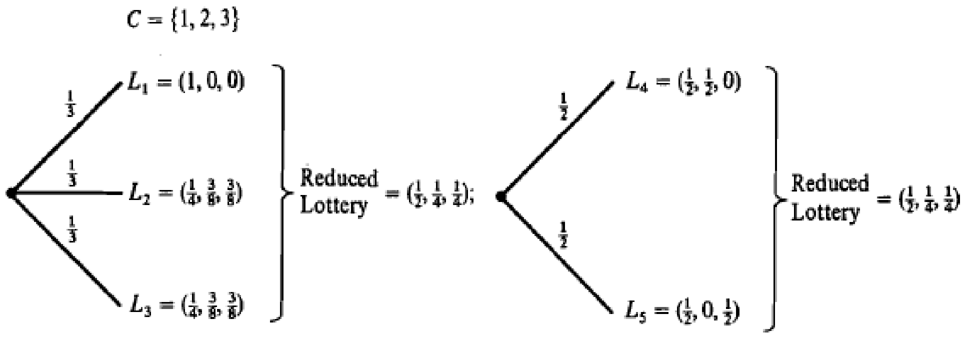

Figure 6.B.3 Two compound lotteries with the same reduced lottery.

for $n = 1, \cdots, N$.[4] Therefore, the reduced lottery L of any compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$ can be obtained by vector addition:

$$L = \alpha_1 L_1 + \cdots + \alpha_K L_K \in \Delta$$

In Figure 6.B.2, two simple lotteries L_1 and L_2 are depicted in the simplex Δ. Also depicted is the reduced lottery $\frac{1}{2}L_1 + \frac{1}{2}L_2$ for the compound lottery $(L_1, L_2; \frac{1}{2}, \frac{1}{2})$ that yields either L_1 or L_2 with a probability of $\frac{1}{2}$ each. This reduced lottery lies at the midpoint of the line segment connecting L_1 and L_2. The linear structure of the space of lotteries is central to the theory of choice under uncertainty, and we exploit it extensively in what follows.

Preferences over Lotteries

Having developed a way to model risky alternatives, we now study the decision maker's preferences over them. The theoretical analysis to follow rest on a basic *consequentialist* premise: We assume that for any risky alternative, only the reduced lottery over final outcomes is of relevance to the decision maker. Whether the probabilities of various outcomes arise as a result of a simple lottery or of a more complex compound lottery has no significance. Figure 6.B.3 exhibits two different compound lotteries that yield the same reduced lottery. Our consequentialist hypothesis requires that the decision maker view these two lotteries as equivalent.

We now pose the decision maker's choice problem in the general framework developed in Chapter 1 (see Section 1.B). In accordance with our consequentialist premise, we take the set of alternatives, denoted here by $\mathscr{L}$, to be *the set of all simple lotteries over the set of outcomes* C. We next assume that the decision maker has a rational preference relation $\succsim$ on $\mathscr{L}$, a complete and transitive relation allowing comparison of any pair of simple lotteries. It should be emphasized that, if anything, the rationality assumption is stronger here than in the theory of choice under certainty discussed in Chapter 1. The more complex the alternatives, the heavier the burden carried by the rationality postulates. In fact, their realism in an uncertainty context has been much debated. However, because we want to concentrate on the properties that are specific to uncertainty, we do not question the rationality assumption further here.

We next introduce two additional assumptions about the decision maker's preferences over lotteries. The most important and controversial is the *independence axiom*. The first, however, is a continuity axiom similar to the one discussed in Section 3.C.

Definition 6.B.3. The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\left\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\right\} \subset [0,1]$$

and

$$\left\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\right\} \subset [0,1]$$

are closed.

In words, continuity means that small changes in probabilities do not change the nature of the ordering between two lotteries. For example, if a "beautiful and uneventful trip by car" is preferred to "staying home," then a mixture of the outcome "beautiful and uneventful trip by car" with a sufficiently small but positive probability of "death by car accident" is still preferred to "staying home." Continuity therefore rules out the case where the decision maker has lexicographic ("safety first") preferences for alternatives with a zero probability of some outcome (in this case, "death by car accident").

As in Chapter 3, the continuity axiom implies the existence of a utility function representing $\succsim$, a function $U : \mathscr{L} \to \mathbb{R}$ such that $L \succsim L'$ if and only if $U(L) \succsim U(L')$. Our second assumption, the independence axiom, will allow us to impose considerably more structure on $U(\cdot)$.[5]

Definition 6.B.4. The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the —it independence axiom if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

In other words, if we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on (is *independent* of) the particular third lottery used.

[5]The independence axiom was first proposed by von Neumann and Morgenstern (1944) as an incidental result in the theory of games.

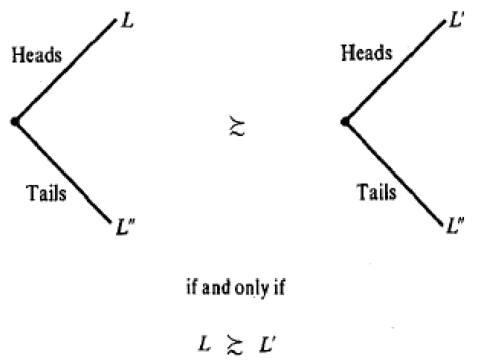

Figure 6.B.4 The independence axiom.

Suppose, for example, that $L \succsim L'$ and $\alpha = \frac{1}{2}$. Then $\frac{1}{2}L + \frac{1}{2}L''$ can be thought of as the compound lottery arising from a coin toss in which the decision maker gets L if heads comes up and L'' if tails does. Similarly, $\frac{1}{2}L' + \frac{1}{2}L''$ would be the coin toss where heads results in L' and tails results in L'' (see Figure 6.B.4). Note that conditional on heads, lottery $\frac{1}{2}L + \frac{1}{2}L''$ is at least as good as lottery $\frac{1}{2}L' + \frac{1}{2}L''$; but conditional on tails, the two compound lotteries give iden!ical results. The independence axiom requires the sensible conclusion that $\frac{1}{2}L + \frac{1}{2}L''$ be at least as good as $\frac{1}{2}L' + \frac{1}{2}L''$.

The independence axiom is at the heart of the theory of choice under uncertainty. It is unlike anything encountered in the formal theory of preference-based choice discussed in Chapter 1 or its applications in Chapters 3 to 5. This is so precisely because it exploits, in a fundamental manner, the structure of uncertainty present in the model. In the theory of consumer demand, for example, there is no reason to believe that a consumer's preferences over various bundles of goods 1 and 2 should be independent of the quantities of the other goods that he will consume. In the present context, however, it is natural to think that a decision maker's preference between two lotteries, say L and L', should determine which of the two he prefers to have as part of a compound lottery *regardless* of the other possible outcome of this compound lottery, say L''. This other outcome L'' should be irrelevant to his choice because, in contrast with the consumer context, he does not consume L or L' together with L'' but, rather, only —it instead of it (if L or L' is the realized outcome).

Exercise 6.B.1 Show that if the preferences $\succsim$ over $\mathscr{L}$ satisfy the independence axiom, then for all $\alpha \in (0,1)$ and $L, L', L'' \in \mathscr{L}$ we have

$$L \succ L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

and

$$L \sim L' \text{ if and only if } \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$$

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

As we will see shortly, the independence axiom is intimately linked to the -representability of preferences over lotteries by a utility function that has an *expected utility form*. Before obtaining that result, we define this property and study,some of its features.

Definition 6.B.5. The utility function $U : \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \cdots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \cdots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N$$

A utility function $U : \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (v.N-M) expected utility function*.

Observe that if we let Ln denote the lottery that yields outcome n with probability one, then $U(L^n) = u_n$. Thus, the term *expected utility* is appropriate because with the v.N-M expected utility form, the utility of a lottery can be thought of as the expected value of the utilities u_n of the N outcomes.

The expression $U(L) = \sum_n u_n p_n$ is a general form for a *linear function in the probabilities* $(p_1, \cdots, p_N)$. This linearity property suggests a useful way to think about the expected utility form.

Proposition 6.B.1. A utility function $U : \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k) \tag{6.B.1}$$

for any K lotteries $L_k \in \mathscr{L}, k = 1, \cdots, K$, and probabilities $(\alpha_1, \cdots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

Proof. Suppose that $U(\cdot)$ satisfies property (6.B.1). We can write any $L = (p_1, \cdots, p_N)$ as a convex combination of the degenerate lotteries $(L^1, \cdots, L^N)$, that is, $L = \sum_n p_n L^n$. We have then $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$. Thus, $U(\cdot)$ has the expected utility form.

In the other direction, suppose that $U(\cdot)$ has the expected utility form, and consider any compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, where $L_k = (p_1^k, \cdots, p_N^k)$. Its reduced lottery is $L' = \sum_k \alpha_k L_k$. Hence,

$$U\left(\sum_k \alpha_k L_k\right) = \sum_n u_n \left(\sum_k \alpha_k p_n^k\right) = \sum_k \alpha_k \left(\sum_n u_n p_n^k\right) = \sum_k \alpha_k U(L_k)$$

Thus, property (6.B.1) is satisfied. *Q.E.D*

The expected utility property is a *cardinal* property of utility functions defined on the space of lotteries. In particular, the result in Proposition 6.B.2 shows that the expected utility form is preserved only by increasing *linear* transformations.

Proposition 6.B.2. Suppose that $U : \mathscr{L} \to \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U} : \mathscr{L} \to \mathbb{R}$ is another v.N-M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proof. Begin by choosing two lotteries $\overline{L}$ and $\underline{L}$ with the property that $\overline{L} \succsim L \succsim \underline{L}$, for all $L \in \mathscr{L}$,[6] If $\overline{L} \sim \underline{L}$, then every utility function is a constant and the result follows immediately. Therefore, we assume from now on that $\overline{L} \succ \underline{L}$.

[6]These best and worst lotteries can be shown to exist. We could, for example, choose a maximizer and a minimizer of the linear, hence continuous, function $U(\cdot)$ on the simplex of probabilities, a compact set.

Note first that if $U(\cdot)$ is a v.N-M expected utility function and $\tilde{U}(L) = \beta U(L) + \gamma$, then

$$\begin{aligned}
\tilde{U}\left(\sum_{k=1}^{K} \alpha_k L_k\right) &= \beta U\left(\sum_{k=1}^{K} \alpha_k L_k\right) + \gamma \\
&= \beta\left[\sum_{k=1}^{K} \alpha_k U(L_k)\right] + \gamma \\
&= \sum_{k=1}^{K} \alpha_k\left[\beta U(L_k) + \gamma\right] \\
&= \sum_{k=1}^{K} \alpha_k \tilde{U}(L_k).
\end{aligned}$$

Since $\tilde{U}(\cdot)$ satisfies property (6.B.1), it has the expected utility form.

For the reverse direction, we want to show that if both $\tilde{U}(\cdot)$ and $U(\cdot)$ have the expected utility form, then constants $\beta > 0$ and γ exist such that $\tilde{U}(L) = \beta U(L) + \gamma$ for all $L \in \mathscr{L}$. To do so, consider any lottery $L \in \mathscr{L}$, and define $\lambda_L \in [0,1]$ by

$$U(L) = \lambda_L U(\overline{L}) + (1 - \lambda_L) U(\underline{L}).$$

Thus

$$\lambda_L = \frac{U(L) - U(\underline{L})}{U(\overline{L}) - U(\underline{L})} \tag{6.B.2}$$

Since $\lambda_L U(\overline{L}) + (1-\lambda_L) U(\underline{L}) = U(\lambda_L \overline{L} + (1-\lambda_L)\underline{L})$ and $U(\cdot)$ represents the preferences $\succsim$, it must be that $L \sim \lambda_L \overline{L} + (1-\lambda_L)\underline{L}$. But if so, then since $\tilde{U}(\cdot)$ is also linear and represents these same preferences, we have

$$\begin{aligned}
\tilde{U}(L) &= \tilde{U}(\lambda_L \overline{L} + (1-\lambda_L)\underline{L}) \\
&= \lambda_L \tilde{U}(\overline{L}) + (1-\lambda_L)\tilde{U}(\underline{L}) \\
&= \lambda_L(\tilde{U}(\overline{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}).
\end{aligned}$$

Substituting for λ_L, from (6.B.2) and rearranging terms yields the conclusion that $\tilde{U}(L) = \beta U(L) + \gamma$, where

$$\beta = \frac{\tilde{U}(\overline{L}) - \tilde{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})}$$

and

$$\gamma = \tilde{U}(\underline{L}) - U(\underline{L})\frac{\tilde{U}(\overline{L}) - \tilde{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})}.$$

This completes the proof *Q.E.D*

A consequence of Proposition 6.B.2 is that for a utility function with the expected utility form, differences of utilities have meaning. For example, if there are four outcomes, the statement "the difference in utility between outcomes 1 and 2 is greater than the difference between outcomes 3 and 4," $u_1 - u_2 > u_3 - u_4$, is equivalent to

$$\frac{1}{2}u_1 + \frac{1}{2}u_4 > \frac{1}{2}u_2 + \frac{1}{2}u_3.$$

Therefore, the statement means that the lottery $L = (\frac{1}{2}, 0, 0, \frac{1}{2})$ is preferred to the lottery $L' = (0, \frac{1}{2}, \frac{1}{2}, 0)$. This ranking of utility differences is preserved by all linear transformations of the v.N-M expected utility function.

Note that if a preference relation $\succsim$ on $\mathscr{L}$ is representable by a utility function $U(\cdot)$ that has the expected utility form, then since a linear utility function is continuous, it follows that $\succsim$ is continuous on $\mathscr{L}$. More importantly, the preference relation $\succsim$ must also satisfy the independence axiom. You are asked to show this in Exercise 6.B.2.

Exercise 6.B.2 Show that if the preference relation $\succsim$ on $\mathscr{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.

The expected utility theorem, the central result of this section, tells us that the converse is also true.

The Expected Utility Theorem

The *expected utility theorem* says that if the decision maker's preferences over lotteries satisfy the continuity and independence axioms, then his preferences are representable by a utility function with the expected utility form. It is the most important result in the theory of choice under uncertainty, and the rest of the book bears witness to its usefulness.

Before stating and proving the result formally, however, it may be helpful to attempt an intuitive understanding of why it is true.

Consider the case where there are only three outcomes. As we have already observed, the continuity axiom insures that preferences on lotteries can be represented by some utility function. Suppose that we represent the indifference map in the simplex, as in Figure 6.B.5. Assume, for simplicity, that we have a conventional map with one-dimensional indifference curves. Because the expected utility form is linear in the probabilities, representability by the expected utility form is equivalent to these indifference curves being straight, parallel lines (you should check this). Figure 6.B.5(a) exhibits an indifference map satisfying these properties. We now argue that these properties are, in fact, consequences of the independence axiom.

Indifference curves are straight lines if, for every pair of lotteries L, L', we have that $L \sim L'$ implies $\alpha L + (1-\alpha)L' \sim L$ for all $\alpha \in [0,1]$. Figure 6.B.5(b) depicts a situation where the indifference curve is not a straight line; we have $L' \sim L$ but $\frac{1}{2}L' + \frac{1}{2}L \succ L$. This is equivalent to saying that

$$\frac{1}{2}L' + \frac{1}{2}L \succ \frac{1}{2}L + \frac{1}{2}L. \tag{6.B.3}$$

But since $L \sim L'$, the independence axiom implies that we must have $\frac{1}{2}L' + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L$ (see Exercise 6.B.1). This contradicts (6.B.3), and so we must conclude that indifference curves are straight lines.

Figure 6.B.5(c) depicts two straight but nonparallel indifference lines. A violation of the independence axiom can be constructed in this case, as indicated in the figure. There we have

$L \succsim L'$ (in fact, $L \sim L'$), but $\frac{1}{3}L + \frac{2}{3}L'' \succsim \frac{1}{3}L' + \frac{2}{3}L''$ does not hold for the lottery L'' shown in the figure. Thus, indifference curves must be parallel, straight lines if preferences satisfy the independence axiom.

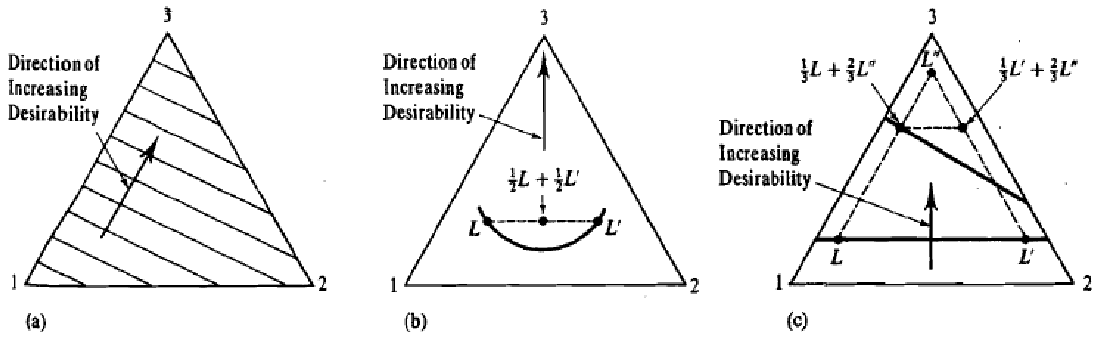

Figure 6.B.5 Geometric explanation of the expected utility theorem. (a) $\succsim$ is representable by a utility function with the expected utility form. (b) Contradiction of the independence axiom. (c) Contradiction of the independence axiom.

Proposition 6.B.3. (*Expected Utility Theorem*) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number un to each outcome $n = 1, \cdots, N$ in such a manner that for any two lotteries $L = (p_1, \cdots, p_N)$ and $L' = (p'_1, \cdots, P'_N)$, we have

$$L \succsim L' \text{ if and only if } \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n. \tag{6.B.4}$$

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Part II

Game Theory

In Part I, we analyzed individual decision making, both in abstract decision problems and in more specific economic settings. Our primary aim was to lay the groundwork for the study of how the simultaneous behavior of many self-interested individuals (including firms) generates economic outcomes in market economies. Most of the remainder of the book is devoted to this task. In Part II, however, we study in a more general way how multiperson interactions can be modeled.

A central feature of multiperson interaction is the potential for the presence of *strategic interdependence*. In our study of individual decision making in Part I, the decision maker faced situations in which her well-being depended only on the choices she made (possibly with some randomness). In contrast, in multiperson situations with strategic interdependence, each agent .recognizes that the payoff she receives (in utility or profits) depends not only on her own actions but also on the actions of *other* individuals. The actions that are best for her to take may depend on actions these other individuals have already taken, on those she expects them to be taking at the same time, and even on future actions that they may take, or decide not to take, as a result of her current actions.

The tool that we use for analyzing settings with strategic interdependence is *noncooperative game theory*. Although the term "game" may seem to undersell the theory's importance, it correctly highlights the theory's central feature: The agents under study are concerned with strategy and winning (in the general sense of utility or profit maximization) in much the same way that players of most parlor games are.

Multiperson economic situations vary greatly in the degree to which strategic interaction is present. In settings of monopoly (where a good is sold by only a single firm; see Section **??**) or of perfect competition (where all agents act as price takers; see Chapter 8 and Part **??**), the nature of strategic interaction is minimal enough that our analysis need not make any formal use of game theory.[1] In other settings, however, such as the analysis of oligopolistic markets (where there is more than one but still not many sellers of a good; see Sections **??** to **??**), the central role of strategic interaction makes game theory indispensable for our analysis.

Part II is divided into three chapters. Chapter 7 provides a short introduction to the basic elements of noncooperative game theory, including a discussion of exactly what a game is, some ways of representing games, and an introduction to a central concept of the theory, a player's *strategy*. Chapter **??** addresses how we can predict outcomes in the special class of games in which all the players move simultaneously, known as *simultaneous-move games*. This restricted focus helps us isolate some central issues while deferring a number of more difficult ones. Chapter **??** studies *dynamic games* in which players' moves may precede one another, and in which some of these more difficult (but also interesting) issues arise.

Note that we have used the modifier *noncooperative* to describe the type of game theory we discuss in Part II. There is another branch of game theory, known as *cooperative game theory*,

[1] However, we could well do so in both cases; see, for example, the proof of existence of competitive equilibrium in Chapter **??**, Appendix **??**. Moreover, we shall stress bow perfect competition can be viewed usefully as a limiting case of oligopolistic strategic interaction; see, for example, Section **??**.

that we do not discuss here. In contrast with noncooperative game theory, the fundamental units of analysis in cooperative theory are groups and subgroups of individuals that are assumed, as a primitive of the theory, to be able to attain particular outcomes for themselves through binding cooperative agreements. Cooperative game theory has played an important role in general equilibrium theory, and we provide a brief introduction to it in Appendix A of Chapter **??**. We should emphasize that the term *noncooperative game theory* does *not* mean that noncooperative theory is incapable of explaining cooperation within groups of individuals. Rather, it focuses on how cooperation may emerge as rational behavior in the absence of an ability to make binding agreements (e.g., see the discussion of repeated interaction among oligopolists in Chapter **??**).

Some excellent recent references for further study of noncooperative game theory are Fudenberg and Tirole (3, 1991), Myerson (7, 1992), and Osborne and Rubinstein (9, 1994), and at a more introductory level Gibbons (4, 1992) and Bimnore (1, 1992). Kreps (5, 1990) provides a very interesting discussion of some of the strengths and weaknesses of the theory. Von Neumann and Morgenstern (8, 1944), Luce and Raiffa (6, 1957), and Schelling (10, 1960) remain classic references.

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Chapter 7
Basic Elements of Noncooperative Games

7.A Introduction

In this chapter, we begin our study of noncooperative game theory by introducing some of its basic building blocks. This material serves as a prelude to our analysis of games in Chapters **??** and **??**.

Section **??** begins with an informal introduction to the concept of a *game*. It describes the four basic elements of any setting of strategic interaction that we must know to specify a game.

In Section **??**, we show how a game can be described by means of what is called its *extensive form representation*. The extensive form representation provides a very rich description of a game, capturing who moves when, what they can do, what they know when it is their turn to move, and the outcomes associated with any collection of actions taken by the individuals playing the game.

In Section **??**, we introduce a central concept of game theory, a player's *strategy*. A player's strategy is a complete contingent plan describing the actions she will take in each conceivable evolution of the game. We then show how the notion of a strategy can be used to derive (much more compact representation of a game, known as its *normal* (or *strategic*) *form representation*.

In Section **??**, we consider the possibility that a player might randomize her choices. This gives rise to the notion of a mixed strategy.

Part III

Market Equilibrium and Market Failure

In Part III, our focus shifts to the fundamental issue of economics: *the organization of production and the allocation of the resulting commodities among consumers*. This fundamental issue can be addressed from two perspectives, one *positive* and the other *normative*.

From a positive (or *descriptive*) perspective, we can investigate the determination of production and consumption under various institutional mechanisms. The institutional arrangement that is our central focus is that of a *market* (or *private ownership*) *economy*. In a market economy, individual consumers have ownership rights to various assets (such as their labor) and are free to trade these assets in the marketplace for other assets or goods. Likewise, firms, which are themselves owned by consumers, decide on their production plan and trade in the market to secure necessary inputs and sell the resulting outputs. Roughly speaking, we can identify a *market equilibrium* as an outcome of a market economy in which each agent in the economy (i.e., each consumer and firm) is doing as well as he can given the actions of all other agents.

In contrast, from a normative (or *prescriptive*) perspective, we can ask what constitutes a *socially optimal* plan of production and consumption (of course, we will need to be more specific about what "socially optimal" means), and we can then examine the extent to which specific institutions, such as a market economy, perform well in this regard.

In Chapter 8, we study *competitive* (or *perfectly competitive*) *market economies* for the first time. These are market economies in which every relevant good is traded in a market at publicly known prices and all agents act as price takers (recall that much of the analysis of individual behavior in Part I was geared to this case). We begin by defining, in a general way, two key concepts: *competitive* (or *Walrasian*) *equilibrium* and *Pareto optimality* (or *Pareto efficiency*). The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any social optimal economic outcome should pass. An economic outcome is said to be Pareto optimal if it is impossible to make some individuals better off without making some other individuals worse off. This concept is a formalization of the idea that there is no waste in society, and it conveniently separates the issue of economic efficiency from more controversial (and political). questions regarding the ideal *distribution* of well-being across individuals.

Chapter 8 then explores these two concepts and the relationships between them in the special context of the *partial equilibrium model*. The partial equilibrium model, which forms the basis for our analysis throughout Part III, offers a considerable analytical simplification; in it, our analysis can be conducted by analyzing a single market (or a small group of related markets) at a time. In this special context, we establish two central results regarding the optimality properties of competitive equilibria, known as the *fundamental theorems of welfare economics*. These can be roughly paraphrased as follows:

> **The First Fundamental Welfare Theorem.** If every relevant good is traded in a market at publicly known prices (i.e., if there is a complete set of markets), and if households and firms act perfectly competitively (i.e., as price takers), then the mar-

ket outcome is Pareto optimal. That is when markets are complete, any competitive equilibrium is necessarily Pareto optimal.

The Second Fundamental Welfare Theorem. If household preferences and firm production sets are convex, there is a complete set of markets with publicly • known prices, and every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.

The first welfare theorem provides a set of conditions under which we can be assured that a market economy will achieve a Pareto optimal result; it is in a sense, the formal expression of Adam Smith's claim about the "invisible hand" of the market. The second welfare theorem goes even further. It states that under the same set of assumptions as the first welfare theorem plus convexity conditions, all Pareto optimal outcomes can in principle be implemented through the market mechanism. That is, a public authority who wishes to implement a particular Pareto optimal outcome (reflecting, say, some political consensus on proper distributional goals) may always do so by appropriately redistributing wealth and then "letting the market work."

In an important sense, the first fundamental welfare theorem establishes the perfectly competitive ca.se as a benchmark for thinking about outcomes in market economies. In particular, any inefficiencies that arise in a market economy, and hence any role for Pareto-improving market intervention, *must* be traceable to a violation of at least one of the assumptions of this theorem.

The remainder of Part III, Chapters **??** to **??**, can be viewed as a development of this theme. In these chapters, we study a number of ways in which actual markets may depart from this perfectly competitive ideal and where, as a result, market equilibria fail to be Pareto optimal, a situation known as *market failure*.

In Chapter **??**, we study *externalities* and *public goods*. In both cases, the actions of one agent directly affect the utility functions or production sets of other agents in the economy. We see there that the presence of these nonmarketed "goods" or "bads" (which violates the complete markets assumption of the first welfare theorem) undermines the Pareto optimality of market equilibrium.

In Chapter **??**, we turn to the study of settings in which some agents in the economy have *market power* and, as a result, fail to act as price takers. Once again, an assumption of the first fundamental welfare theorem fails to hold, and market equilibria fail to be Pareto optimal as a result.

In Chapters **??** and **??**, we consider situations in which an *asymmetry of information* exists among market participants. The complete markets assumption of the first welfare theorem implicitly requires that the characteristics of traded commodities be observable by all market participants because, without this observability, distinct markets cannot exist for commodities that have different characteristics. Chapter **??** focuses on the case in which asymmetric information exists between agents at the time of contracting. Our discussion highlights several phenomena-*adverse selection*, *signaling*, and *screening*-that can arise as a result of this informational imperfection, and the welfare loss that it causes. Chapter **??** in contrast, investigates the case of postcontractual asymmetric information, a problem that leads us to the study of the *principal-agent model*. Here, too, the presence of asymmetric information prevents trade of all relevant commodities and can lead market outcomes to be Pareto inefficient.

We rely extensively in some places in Part III on the tools that we developed in Parts I and II. This is particularly true in Chapter 8, where we use material developed in Part I, and Chapters **??** and **??**, where we use the game-theoretic tools developed in Part II.

A much more complete and general study of competitive market economies and the fun!iamental welfare theorems is reserved for Part **??**.

Chapter 8
Competitive Markets

8.A Introduction

In this chapter, we consider, for the first time, an entire economy in which consumers and firms interact through markets. The chapter has two principal goals: first, to formally introduce and study two key concepts, the notions of *Pareto optimality* and *competitive equilibrium*, and second, to develop a somewhat special but analytically very tractable context for the study of market equilibrium, the *partial equilibrium model*.

We begin in Section lO.B by presenting the notions of a *Pareto optimal* (or *Pareto efficient*) *allocation* and of a *competitive* (or *Walrasian*) *equilibrium* in a general setting.

Starting in Section **??**, we narrow our focus to the partial equilibrium context. The partial equilibrium approach, which originated in Marshall (1920), envisions the market for a single good (or group of goods) for which each consumer's expenditure constitutes only a small portion of his overall budget. When this is so, it is reasonable to assume that changes in the market for this good will leave the prices of all other commodities approximately unaffected and that there will be, in addition, negligible wealth effects in the market under study. We capture these features in the simplest possible way by considering a two-good model in which the expenditure on all commodities other than that under consideration is treated as a single composite commodity (called the *numeraire* commodity), and in which consumers' utility functions take a quasilinear form with respect to this numeraire. Our study of the competitive equilibria of this simple model lends itself to extensive demand-and-supply graphical analysis. We also discuss how to determine the comparative statics effects that arise from exogenous changes in the market environment. As an illustration, we consider the effects on market equilibrium arising from the introduction of a distortionary commodity tax.

In Section **??**, we analyze the properties of Pareto optimal allocations in the partial equilibrium model. Most significantly, we establish for this special context the validity of the *fundamental theorems of welfare economics*: Competitive equilibrium allocations are necessarily Pareto optimal, and any Pareto optimal allocation can be achieved as a competitive equilibrium if appropriate lump-sum transfers are made. As we noted in the introduction to Part III, these results identify an important benchmark case in which market equilibria yield desirable economic outcomes. At the same time, they provide a framework for identifying situations of market failure, such as those we study in Chapters **??** to **??**.

In Section **??**, we consider the measurement of welfare changes in the partial equilibrium context. We show that these can be represented by areas between properly defined demand and supply curves. As an application, we examine the deadweight loss of distortionary taxation.

Section **??** contemplates settings characterized by *free entry*, that is, settings in which all potential firms have access to the most efficient technology and may enter and exit markets in response to the profit opportunities they present. We define a notion of *long-run competitive equilibrium* and then use it to distinguish between long run and short-run comparative static effects in response to changes in market conditions.

In Section **??**, we provide a more extended discussion of the use of Partial equilibrium analysis in economic modeling.

The material covered in this chapter traces its roots far back in economic thought. An excellent source for further reading is Stigler (1987). We should emphasize that the analysis of competitive equilibrium and Pareto optimality presented here is very much a first pass. In Part **??** we return to the topic for a more complete and general investigation; many additional references will be given there.